

DYNAMIC ASYMPTOTIC DIMENSION AND K -THEORY OF BANACH CROSSED PRODUCT ALGEBRAS

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ABSTRACT. We apply controlled (or quantitative) K -theory to prove that a certain L_p assembly map is an isomorphism for $p \in (1, \infty)$ when a countable discrete group Γ acts with finite dynamic asymptotic dimension on a compact Hausdorff space X . When $p = 2$, this is a model for the Baum-Connes assembly map for Γ with coefficients in $C(X)$, and was shown to be an isomorphism by Guentner, Willett, and Yu. As a consequence, we see that the K -theory of the L_p reduced crossed product is independent of $p \in (1, \infty)$ under the assumption of finite dynamic asymptotic dimension.

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1. INTRODUCTION

Notions of dimension abound in mathematics, and they give us quantitative measures of the sizes of various mathematical objects in a broad sense. In some instances, one wishes to know the exact dimension while in other instances, one just wishes to determine finiteness of the dimension. Finiteness of various dimensions has been considered in connection with central problems in the theory of C^* -algebras and in noncommutative geometry.

Date: November 29, 2016.

2010 *Mathematics Subject Classification.* 19K56, 46L80, 46L85 47L10, 37B05.

Key words and phrases. Quantitative operator K -theory, L^p operator algebra, L^p crossed product, Baum-Connes conjecture.

For instance, finiteness of nuclear dimension plays a crucial role in the classification of C^* -algebras [4, 5, 17], while finiteness of asymptotic dimension or of dynamic asymptotic dimension has featured in work on the Baum-Connes conjecture [20, 6]. In these works on the Baum-Connes conjecture, finiteness of dimension allows one to apply cutting-and-pasting techniques (i.e., Mayer-Vietoris sequences) a finite number of times to compute the K -theory of a certain C^* -algebra. In this paper, we will consider dynamic asymptotic dimension and the implication of its finiteness on an L_p version of the Baum-Connes conjecture with coefficients.

Dynamic asymptotic dimension is a property of topological dynamical systems introduced by Guentner, Willett, and Yu in [7] for discrete groups acting by homeomorphisms on locally compact Hausdorff spaces, and it can be defined as follows.

Definition 1.1. [7] *An action of a countable discrete group Γ on a locally compact Hausdorff space X has dynamic asymptotic dimension d if d is the smallest natural number with the following property: for any compact subset K of X and finite subset E of Γ , there are open subsets U_0, \dots, U_d of X that cover K such that for each $i \in \{0, \dots, d\}$, the set*

$$\left\{ g \in \Gamma : \begin{array}{l} \text{there exist } x \in U_i \text{ and } g_n, \dots, g_1 \in E \text{ such} \\ \text{that } g = g_n \cdots g_1 \text{ and } g_k \cdots g_1 x \in U_i \text{ for all} \\ k \in \{1, \dots, n\} \end{array} \right\}$$

is finite.

One thinks of finite dynamic asymptotic dimension as a condition that allows one to break up the action into at most a certain number of parts whenever we are given a finite subset of the group, and such that on each part the action is fairly simple if we restrict our attention to the given finite subset. The main motivation of the authors of [7] was the implications for K -theory of associated C^* -algebras, such as crossed products, and thus for manifold topology. They investigated some connections with controlled topology, coarse geometry, and structure of C^* -algebras (in particular nuclear dimension). They also defined the dynamic asymptotic dimension of locally compact Hausdorff étale groupoids, and showed that an action has dynamic asymptotic dimension d if and only if the corresponding transformation groupoid has dynamic asymptotic dimension d . Indeed, the groupoid point of view is useful if one wishes to decompose algebras associated with such actions in order to apply cutting-and-pasting techniques.

In [6], the same authors considered a model for the Baum-Connes assembly map for an action based on Yu's localization algebras [19] and Roe algebras, and they proved the following result.

Theorem 1.2. [6] *Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X . Then the Baum-Connes conjecture holds for Γ with coefficients in $C(X)$.*

The hypothesis of having finite dynamic asymptotic dimension includes many interesting actions. For example, it was shown in [7] that all free minimal \mathbb{Z} -actions on compact spaces (such as irrational rotation of the circle) have dynamic asymptotic dimension one, and that groups with finite asymptotic dimension act with finite dynamic asymptotic dimension on some compact space.

Although this result follows from earlier work of Tu [18] on the Baum-Connes conjecture for amenable groupoids, the proof given in [6] is completely different, and in some sense more direct. In fact, the proof is very much inspired by Yu's proof of the coarse Baum-Connes conjecture for spaces with finite asymptotic dimension in [20]. The main tool in both cases is a controlled Mayer-Vietoris sequence, which is part of a framework of controlled (or quantitative) K -theory for C^* -algebras developed by Yu together with Oyono-Oyono in [11] and [12]. Finiteness of the appropriate notion of dimension allows one to apply the Mayer-Vietoris argument a finite number of times to arrive at the K -theory of the algebra in question.

In our earlier work [2], we have extended the framework of controlled K -theory to a larger class of Banach algebras, so that it can be applied to algebras of bounded linear operators on L_p spaces. Our goal in this paper is to consider the L_p analog of the assembly map in [6], and use our extended framework of controlled K -theory to show that this assembly map is an isomorphism under the assumption of finite dynamic asymptotic dimension. In fact, one sees that the techniques and proofs in [6] carry over to our setting with minor adjustments. In the appendix of [6], the authors show that their model for the Baum-Connes assembly map agrees with the one stated in terms of Kasparov's KK -theory [1]. In order to state our main result, let us first recall the usual Baum-Connes conjecture with coefficients.

Let A be a C^* -algebra and let a countable discrete group Γ act on A by $*$ -automorphisms. One may then form the reduced crossed product C^* -algebra $A \rtimes_\lambda \Gamma$. The usual Baum-Connes conjecture with coefficients posits that a certain homomorphism

$$\mu : K_*^\Gamma(\underline{E}\Gamma; A) \rightarrow K_*(A \rtimes_\lambda \Gamma)$$

is an isomorphism [1], where the left-hand side is the equivariant K -homology of the classifying space for proper Γ -actions with coefficients in A , and the right-hand side is the K -theory of the reduced crossed product C^* -algebra. We will consider a particular model for $\underline{E}\Gamma$, namely $\bigcup_{s \geq 0} P_s(\Gamma)$ equipped with the ℓ_1 metric [1, Section 2], where $P_s(\Gamma)$ is the Rips complex of Γ at scale s , i.e., it is the simplicial complex with vertex set Γ , and where a finite subset $E \subset \Gamma$ spans a simplex if and only if $d(g, h) \leq s$ for all $g, h \in E$. Here we assume that Γ is equipped with a proper length function and d is the associated metric. One may then reformulate the Baum-Connes map as

$$\lim_{s \rightarrow \infty} K_*(C_L^*(P_s(\Gamma); A)) \xrightarrow{\epsilon_0} \lim_{s \rightarrow \infty} K_*(C^*(P_s(\Gamma); A)) \cong K_*(A \rtimes_\lambda \Gamma),$$

where $C_L^*(P_s(\Gamma); A)$ is Yu's localization algebra [19] with coefficients in A , $C^*(P_s(\Gamma); A)$ is the equivariant Roe algebra with coefficients in A , and ϵ_0 is (induced by) the evaluation-at-zero map. The fact that K -homology can be identified with the K -theory of the localization algebra was shown for finite-dimensional simplicial complexes in [19], and in full generality in [15]. The fact that the equivariant Roe algebra with coefficients is stably isomorphic to the reduced crossed product forms the basis for the coarse-geometric approach to the Baum-Connes conjecture with coefficients (see [16] for the case without coefficients).

Now let A be a closed subalgebra of $B(L_p(Z, \mu))$ for some measure space (Z, μ) and $p \in (1, \infty)$. We refer to such algebras as L_p operator algebras. Let Γ be a countable discrete group acting on A by isometric automorphisms. Define $A\Gamma = \{\sum_{g \in \Gamma} a_g g : a_g \in A\}$ with the product given by

$$\left(\sum_{g \in \Gamma} a_g g\right) \left(\sum_{h \in \Gamma} b_h h\right) = \sum_{g, h \in \Gamma} a_g \alpha_g(b_h) gh,$$

where α denotes the Γ -action on A . There is a natural faithful representation of $A\Gamma$ on $\ell_p(\Gamma, L_p(Z, \mu))$ given by

$$\begin{aligned} (a\xi)(h) &= \alpha_{g^{-1}}(a)\xi(h), \\ (g\xi)(h) &= \xi(g^{-1}h) \end{aligned}$$

for $a \in A$, $g, h \in \Gamma$, and $\xi \in \ell_p(\Gamma, L_p(Z, \mu))$. We then define the L_p reduced crossed product $A \rtimes_{\lambda, p} \Gamma$ to be the operator norm closure of $A\Gamma$ in $B(\ell_p(\Gamma, L_p(Z, \mu)))$.

We can formulate the L_p Baum-Connes conjecture with coefficients by replacing the reduced crossed product C^* -algebra by the L_p reduced crossed product, and also considering L_p versions of Roe algebras and localization algebras, so that the map in question essentially becomes

$$\lim_{s \rightarrow \infty} K_*(C_L^*(P_s(\Gamma); A)) \rightarrow K_*(A \rtimes_{\lambda, p} \Gamma),$$

and this map is posited to be an isomorphism. Here we note that one can show that the left-hand side is independent of p by following the arguments in [19], and the L_p version of the equivariant Roe algebra is stably isomorphic to the L_p reduced crossed product by the same argument as in the C^* -algebra case.

Our main result may then be stated as follows.

Theorem 1.3. *(c.f. Theorem 5.22) Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X . Then the L_p Baum-Connes conjecture holds for Γ with coefficients in $C(X)$ for $p \in (1, \infty)$.*

Since the left-hand side of the map is independent of p , we have the following corollary, which gives a partial answer to [13, Problem 11.2].

Corollary 1.4. *Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X . Then the K -theory of the L_p reduced crossed product is independent of p for $p \in (1, \infty)$.*

In the L_p setting, we note that Kasparov and Yu have some (yet unpublished) work on the L_p Baum-Connes conjecture [8]. We also remark that at the moment, there seems to be substantial difficulty in carrying over other approaches to the usual Baum-Connes conjecture, such as the Dirac-dual Dirac method, to the L_p setting.

The question of whether the K -theory of the L_p reduced crossed product depends on p remains open in general. In the case where X is a point with the trivial Γ -action, one gets the L_p reduced group algebra. When Γ is hyperbolic or amenable, the K -theory of this algebra was shown to be independent of p [8, 10]. The significance of this question is that sometimes the K -theory of these algebras may be more computable for large p so if the K -theory of these algebras is independent of p , then we get in particular a computation of the K -theory of the respective C^* -algebras.

This paper is organized as follows: In section 2, we define certain L_p Roe algebras and localization algebras associated to an action of a countable discrete group on a compact Hausdorff space, and we define an L_p assembly map in terms of the K -theory of these algebras. In the case $p = 2$, these are exactly the algebras and map considered in [6]. In section 3, we associate subalgebras of the algebras introduced in section 2 to subgroupoids of the transformation groupoid given by the action, and recall some facts about dynamic asymptotic dimension in terms of groupoids. In section 4, we recall some definitions and facts from the framework of controlled K -theory that we developed in [2]. Finally, in section 5, we prove our main result using an induction argument with a controlled Mayer-Vietoris sequence.

Acknowledgements. We would like to thank Guoliang Yu for his patient guidance and support throughout this project, and also Rufus Willett for explaining various aspects of their work in the C^* -algebraic setting.

2. AN L_p ASSEMBLY MAP

Throughout this section, Γ will be a countable discrete group acting on a compact Hausdorff space X by homeomorphisms. The action will be denoted by $\Gamma \curvearrowright X$. We also assume that Γ is equipped with a proper length function $l : \Gamma \rightarrow \mathbb{N}$ and the associated right invariant metric. We will define an assembly map in terms of L_p versions of localization algebras and Roe algebras, where $p \in (1, \infty)$. In the case $p = 2$, we recover (a model of) the Baum-Connes assembly map for Γ with coefficients in $C(X)$ considered in [6].

Definition 2.1. *Let $s \geq 0$. The Rips complex of Γ at scale s , denoted $P_s(\Gamma)$, is the simplicial complex with vertex set Γ , and where a finite subset $E \subset \Gamma$ spans a simplex if and only if $d(g, h) \leq s$ for all $g, h \in E$.*

Points in $P_s(\Gamma)$ can be written as formal linear combinations $\sum_{g \in \Gamma} t_g g$, where $t_g \in [0, 1]$ for each g and $\sum_{g \in \Gamma} t_g = 1$. We equip $P_s(\Gamma)$ with the ℓ_1 metric, i.e., $d(\sum_{g \in \Gamma} t_g g, \sum_{g \in \Gamma} s_g g) = \sum_{g \in \Gamma} |t_g - s_g|$.

The barycentric coordinates on $P_s(\Gamma)$ are the continuous functions

$$t_g : P_s(\Gamma) \rightarrow [0, 1]$$

uniquely determined by the condition $z = \sum_{g \in \Gamma} t_g(z)g$ for all $z \in P_s(\Gamma)$.

By assumption of properness of the length function on Γ , one sees that $P_s(\Gamma)$ is finite-dimensional and locally compact. Also, the right translation action of Γ on itself extends to a right action of Γ on $P_s(\Gamma)$ by isometric simplicial automorphisms.

In the usual setting of the Baum-Connes conjecture, one considers Hilbert spaces and C^* -algebras encoding the large scale geometry of Γ and the topology of $P_s(\Gamma)$. We will replace these Hilbert spaces by L_p spaces, thereby obtaining L_p operator algebras instead of C^* -algebras.

First, we recall some facts about L_p tensor products. Details can be found in [3, Chapter 7].

For $p \in [1, \infty)$, there is a tensor product of L_p spaces such that we have a canonical isometric isomorphism $L_p(X, \mu) \otimes L_p(Y, \nu) \cong L_p(X \times Y, \mu \times \nu)$, which identifies, for every $\xi \in L_p(X, \mu)$ and $\eta \in L_p(Y, \nu)$, the element $\xi \otimes \eta$ with the function $(x, y) \mapsto \xi(x)\eta(y)$ on $X \times Y$. Moreover,

- Under the identification above, the linear span of all $\xi \otimes \eta$ is dense in $L_p(X \times Y, \mu \times \nu)$.
- $\|\xi \otimes \eta\|_p = \|\xi\|_p \|\eta\|_p$ for all $\xi \in L_p(X, \mu)$ and $\eta \in L_p(Y, \nu)$.
- The tensor product is commutative and associative.
- If $a \in B(L_p(X_1, \mu_1), L_p(X_2, \mu_2))$ and $b \in B(L_p(Y_1, \nu_1), L_p(Y_2, \nu_2))$, then there exists a unique

$$c \in B(L_p(X_1 \times Y_1, \mu_1 \times \nu_1), L_p(X_2 \times Y_2, \mu_2 \times \nu_2))$$

such that under the identification above, $c(\xi \otimes \eta) = a(\xi)b(\eta)$ for all $\xi \in L_p(X_1, \mu_1)$ and $\eta \in L_p(Y_1, \nu_1)$. We will denote this operator by $a \otimes b$. Moreover, $\|a \otimes b\| = \|a\| \|b\|$.

- The tensor product of operators is associative, bilinear, and satisfies $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Definition 2.2. For $s \geq 0$, define

$$Z_s = \left\{ \sum_{g \in \Gamma} t_g g \in P_s(\Gamma) : t_g \in \mathbb{Q} \text{ for all } g \in \Gamma \right\}.$$

Note that Z_s is a Γ -invariant subset of $P_s(\Gamma)$.

Define

$$E_s = \ell_p(Z_s) \otimes \ell_p(X) \otimes \ell_p \otimes \ell_p(\Gamma),$$

and equip E_s with the isometric Γ -action given by

$$u_g \cdot (\delta_z \otimes \delta_x \otimes \eta \otimes \delta_h) = \delta_{zg^{-1}} \otimes \delta_{gx} \otimes \eta \otimes \delta_{gh}$$

for $z \in Z_s$, $x \in X$, $\eta \in \ell_p$, and $g, h \in \Gamma$.

Note that we have a canonical isometric isomorphism

$$E_s = \ell_p(Z_s \times X, \ell_p \otimes \ell_p(\Gamma)).$$

Also note that if $s_0 \leq s$, then $P_{s_0}(\Gamma)$ identifies equivariantly and isometrically with a subcomplex of $P_s(\Gamma)$, and $Z_{s_0} \subset Z_s$. Hence we have a canonical equivariant isometric inclusion $E_{s_0} \subset E_s$.

We will write \mathcal{K}_Γ for the algebra of compact operators on $\ell_p \otimes \ell_p(\Gamma) \cong \ell_p(\mathbb{N} \times \Gamma)$ equipped with the Γ -action induced by the tensor product of the trivial action on ℓ_p and the left regular representation on $\ell_p(\Gamma)$. We also equip the algebra $C(X) \otimes \mathcal{K}_\Gamma$ with the diagonal action of Γ . Note that the natural faithful representation of $C(X) \otimes \mathcal{K}_\Gamma$ on $\ell_p(X) \otimes \ell_p \otimes \ell_p(\Gamma)$ is covariant for the representation defined by tensoring the natural action on $\ell_p(X)$, the trivial representation on ℓ_p , and the regular representation on $\ell_p(\Gamma)$.

Now we can define the L_p operator algebras that will feature in our assembly map.

Definition 2.3. Let T be a bounded linear operator on E_s , which we may think of as a $Z_s \times Z_s$ -indexed matrix $T = (T_{y,z})$ with

$$T_{y,z} \in B(\ell_p(X) \otimes \ell_p \otimes \ell_p(\Gamma))$$

for each $y, z \in Z_s$.

- (i) T is Γ -invariant if $u_g T u_g^{-1} = T$ for all $g \in \Gamma$.
- (ii) The Rips-propagation of T is $\sup\{d_{P_s(\Gamma)}(y, z) : T_{y,z} \neq 0\}$.
- (iii) The Γ -propagation of T , denoted by $\text{prop}_\Gamma(T)$, is $\sup\{d_\Gamma(g, h) : T_{y,z} \neq 0 \text{ for some } y, z \in Z_s \text{ with } t_g(y) \neq 0 \text{ and } t_h(z) \neq 0\}$.

- (iv) T is X -locally compact if $T_{y,z} \in C(X) \otimes \mathcal{K}_\Gamma$ for all $y, z \in Z_s$, and if for any compact subset $F \subset P_s(\Gamma)$, the set

$$\{(y, z) \in F \times F : T_{y,z} \neq 0\}$$

is finite.

Definition 2.4. Let $\mathbb{C}[\Gamma \curvearrowright X; s]$ denote the algebra of all Γ -invariant, X -locally compact operators on E_s with finite Γ -propagation.

Let $C^{*,p}(\Gamma \curvearrowright X; s)$ denote the closure of $\mathbb{C}[\Gamma \curvearrowright X; s]$ with respect to the operator norm on E_s . We will call $C^{*,p}(\Gamma \curvearrowright X; s)$ the L_p Roe algebra of $\Gamma \curvearrowright X$ at scale s .

We leave the reader to verify that $\mathbb{C}[\Gamma \curvearrowright X; s]$ is indeed an algebra. We will always regard the algebras above as concretely represented on E_s , and we will often think of elements of $C^{*,p}(\Gamma \curvearrowright X; s)$ as matrices $(T_{y,z})_{y,z \in Z_s}$ with entries being continuous equivariant functions $T_{y,z} : X \rightarrow \mathcal{K}_\Gamma$ having additional properties.

Definition 2.5. Let $\mathbb{C}_L[\Gamma \curvearrowright X; s]$ denote the algebra of all bounded, uniformly continuous functions $a : [0, \infty) \rightarrow \mathbb{C}[\Gamma \curvearrowright X; s]$ such that the Γ -propagation of $a(t)$ is uniformly finite as t varies, and such that the Rips-propagation of $a(t)$ tends to zero as $t \rightarrow \infty$.

Let $C_L^{*,p}(\Gamma \curvearrowright X; s)$ denote the completion of $\mathbb{C}_L[\Gamma \curvearrowright X; s]$ with respect to the norm

$$\|a\| := \sup_t \|a(t)\|_{C^{*,p}(\Gamma \curvearrowright X; s)}.$$

We will call $C_L^{*,p}(\Gamma \curvearrowright X; s)$ the L_p localization algebra of $\Gamma \curvearrowright X$ at scale s .

We will regard these algebras as concretely represented on $L_p[0, \infty) \otimes E_s$. Elements of $C_L^{*,p}(\Gamma \curvearrowright X; s)$ can be regarded as bounded, uniformly continuous functions $a : [0, \infty) \rightarrow C^{*,p}(\Gamma \curvearrowright X; s)$ having additional properties.

Now consider the evaluation-at-zero homomorphism

$$\epsilon_0 : C_L^{*,p}(\Gamma \curvearrowright X; s) \rightarrow C^{*,p}(\Gamma \curvearrowright X; s),$$

which induces a homomorphism

$$\epsilon_0 : K_*(C_L^{*,p}(\Gamma \curvearrowright X; s)) \rightarrow K_*(C^{*,p}(\Gamma \curvearrowright X; s)).$$

If $s_0 \leq s$, then the equivariant isometric inclusion $E_{s_0} \subset E_s$ allows us to regard $\mathbb{C}[\Gamma \curvearrowright X; s_0]$ as a subalgebra of $\mathbb{C}[\Gamma \curvearrowright X; s]$. We then regard $C^{*,p}(\Gamma \curvearrowright X; s_0)$ (resp. $C_L^{*,p}(\Gamma \curvearrowright X; s_0)$) as a subalgebra of $C^{*,p}(\Gamma \curvearrowright X; s)$ (resp. $C_L^{*,p}(\Gamma \curvearrowright X; s)$). Thus there are directed systems of inclusions of L_p operator algebras $(C^{*,p}(\Gamma \curvearrowright X; s))_{s \geq 0}$ and $(C_L^{*,p}(\Gamma \curvearrowright X; s))_{s \geq 0}$, and the evaluation-at-zero maps above are compatible with these inclusions.

Definition 2.6. The L_p assembly map for $\Gamma \curvearrowright X$ is the direct limit

$$\epsilon_0 : \lim_{s \rightarrow \infty} K_*(C_L^{*,p}(\Gamma \curvearrowright X; s)) \rightarrow \lim_{s \rightarrow \infty} K_*(C^{*,p}(\Gamma \curvearrowright X; s)).$$

For most of the rest of this paper, we will work with the kernel of this L_p assembly map.

Definition 2.7. Let $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$ be the subalgebra of $C_L^{*,p}(\Gamma \curvearrowright X; s)$ consisting of functions a such that $a(0) = 0$. We will call $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$ the L_p obstruction algebra of $\Gamma \curvearrowright X$ at scale s .

Lemma 2.8. The L_p assembly map for $\Gamma \curvearrowright X$ is an isomorphism if and only if

$$\lim_{s \rightarrow \infty} K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)) = 0.$$

Proof. We have a short exact sequence

$$0 \rightarrow C_{L,0}^{*,p}(\Gamma \curvearrowright X; s) \rightarrow C_L^{*,p}(\Gamma \curvearrowright X; s) \rightarrow C^{*,p}(\Gamma \curvearrowright X; s) \rightarrow 0,$$

which induces the usual six-term exact sequence in K -theory. The lemma then follows from continuity of K -theory under direct limits, and the preservation of exact sequences under direct limits of abelian groups. \square

Our goal in this paper will be to show that if $\Gamma \curvearrowright X$ has finite dynamic asymptotic dimension, then $\lim_{s \rightarrow \infty} K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)) = 0$, and thus the L_p assembly map is an isomorphism.

3. GROUPOIDS AND DYNAMIC ASYMPTOTIC DIMENSION

For the Mayer-Vietoris argument that we will use in the proof of our main result, it is more convenient to use the language of groupoids instead of group actions. In this section, we consider the transformation groupoid associated to a group action, its subgroupoids, and associated algebras. We also recall the definition of dynamic asymptotic dimension in terms of groupoids.

Definition 3.1. *The transformation groupoid $\Gamma \ltimes X$ associated to $\Gamma \curvearrowright X$ is $\{(gx, g, x) : g \in \Gamma, x \in X\}$ topologized such that the projection $\Gamma \ltimes X \rightarrow \Gamma \times X$ onto the second and third factors is a homeomorphism, and equipped with the following additional structure:*

- (i) *A pair $((hy, h, y), (gx, g, x))$ of elements in $\Gamma \ltimes X$ is said to be composable if $y = gx$. In this case, their product is defined by*

$$(hgx, h, gx)(gx, g, x) = (hgx, hg, x).$$

- (ii) *The inverse of an element $(gx, g, x) \in \Gamma \ltimes X$ is*

$$(gx, g, x)^{-1} = (x, g^{-1}, gx).$$

- (iii) *The units of $\Gamma \ltimes X$ are the elements of the clopen subspace*

$$G^{(0)} = \{(x, e, x) : x \in X\},$$

where e is the identity in Γ . We refer to $G^{(0)}$ as the unit space of $\Gamma \ltimes X$.

Definition 3.2. *Let $s \geq 0$, and let $P_s(\Gamma)$ be the associated Rips complex of Γ . The support of $z = \sum_{g \in \Gamma} t_g(z)g \in P_s(\Gamma)$ is the finite subset*

$$\text{supp}(z) = \{g \in \Gamma : t_g(z) \neq 0\}.$$

The support of $T = (T_{y,z})_{y,z \in Z_s} \in C^{,p}(\Gamma \curvearrowright X; s)$ is*

$$\text{supp}(T) = \left\{ (gx, gh^{-1}, hx) \in \Gamma \ltimes X : \begin{array}{l} \text{there exist } y, z \in P_s(\Gamma) \text{ with} \\ T_{y,z}(x) \neq 0, \ g \in \text{supp}(y), \text{ and} \\ h \in \text{supp}(z) \end{array} \right\}$$

With this definition, one sees that

$$\text{prop}_\Gamma(T) = \sup\{l(gh^{-1}) : (gx, gh^{-1}, hx) \in \text{supp}(T) \text{ for some } x \in X\}.$$

Given two subsets $A, B \subset \Gamma \ltimes X$, we write AB for

$$\{ab : a \in A, b \in B, (a, b) \text{ is composable}\}.$$

With this notation, the following lemma says that supports of operators in $C^{*,p}(\Gamma \curvearrowright X; s)$ behave as expected under composition of operators.

Lemma 3.3. *Let $S, T \in C^{*,p}(\Gamma \curvearrowright X; s)$. Then $\text{supp}(ST) \subset \text{supp}(S)\text{supp}(T)$.*

Proof. Suppose that $(gx, gh^{-1}, hx) \in \text{supp}(ST)$. Then there are $y, z \in P_s(\Gamma)$ such that $(ST)_{y,z}(x) \neq 0$, $g \in \text{supp}(y)$, and $h \in \text{supp}(z)$. Thus there is $w \in P_s(\Gamma)$ such that $S_{y,w}(x) \neq 0$ and $T_{w,z}(x) \neq 0$. If $k \in \text{supp}(w)$, then $(gx, gk^{-1}, kx) \in \text{supp}(S)$ and $(kx, kh^{-1}, hx) \in \text{supp}(T)$, so $(gx, gh^{-1}, hx) = (gx, gk^{-1}, kx)(kx, kh^{-1}, hx) \in \text{supp}(S)\text{supp}(T)$. \square

Definition 3.4. Let $\Gamma \ltimes X$ be the transformation groupoid associated to $\Gamma \curvearrowright X$. A subgroupoid of $\Gamma \ltimes X$ is a subset $G \subset \Gamma \ltimes X$ that is closed under composition, taking inverses, and units, i.e.,

- (i) If (hgx, h, gx) and (gx, g, x) are in G , then so is (hgx, hg, x) .
- (ii) If $(gx, g, x) \in G$, then $(gx, g, x)^{-1} \in G$.
- (iii) If $(gx, g, x) \in G$, then $(x, e, x) \in G$ and $(gx, e, gx) \in G$, where e is the identity in Γ .

A subgroupoid is equipped with the subspace topology from $\Gamma \ltimes X$.

Subgroupoids of $\Gamma \ltimes X$ give rise to subalgebras of the Roe algebra, localization algebra, and obstruction algebra that we defined in the previous section.

Lemma 3.5. Let G be an open subgroupoid of $\Gamma \ltimes X$. Define $\mathbb{C}[G; s]$ to be the subspace of $\mathbb{C}[\Gamma \curvearrowright X; s]$ consisting of all operators T with support contained in a compact subset of G . Then $\mathbb{C}[G; s]$ is a subalgebra of $\mathbb{C}[\Gamma \curvearrowright X; s]$.

Proof. Given Lemma 3.3, it suffices to show that if A and B are two relatively compact subsets of G , then so is AB . We refer the reader to [7, Lemma 5.2] for the proof of this statement. \square

Definition 3.6. Let G be an open subgroupoid of $\Gamma \ltimes X$. Let $\mathbb{C}_L[G; s]$ denote the subalgebra of $\mathbb{C}_L[\Gamma \curvearrowright X; s]$ consisting of functions a such that $\bigcup_{t \in [0, \infty)} \text{supp}(a(t))$ has compact closure in G .

Let $\mathbb{C}_{L,0}[G; s]$ denote the ideal of $\mathbb{C}_L[G; s]$ consisting of functions a such that $a(0) = 0$.

Let $C^{*,p}(G; s)$, $C_L^{*,p}(G; s)$, and $C_{L,0}^{*,p}(G; s)$ denote the respective closures of $\mathbb{C}[G; s]$, $\mathbb{C}_L[G; s]$, and $\mathbb{C}_{L,0}[G; s]$ in $C^{*,p}(\Gamma \curvearrowright X; s)$, $C_L^{*,p}(\Gamma \curvearrowright X; s)$, and $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$.

Since we will be working mostly with the obstruction algebras, we introduce the following shorthand notation for these algebras. We also need to construct filtrations on these algebras so as to apply our controlled K -theory to them later.

Definition 3.7. Let G be an open subgroupoid of $\Gamma \ltimes X$, and let $s \geq 0$. Define $A^s(G)$ to be $C_{L,0}^{*,p}(G; s)$. For $r \geq 0$, define

$$A^s(G)_r = \{a \in \mathbb{C}_{L,0}[G; s] : \text{prop}_\Gamma(a(t)) \leq r \text{ for all } t\},$$

which is a linear subspace of $A^s(G)$.

When $G = \Gamma \ltimes X$, we will simply write A^s and A_r^s .

Lemma 3.8. *Let G be an open subgroupoid of $\Gamma \ltimes X$, and let $s \geq 0$. Then the family $(A^s(G))_{r \geq 0}$ of subspaces of $A^s(G)$ satisfies:*

- (i) *if $r_1 \leq r_2$, then $A^s(G)_{r_1} \subset A^s(G)_{r_2}$;*
- (ii) *$A^s(G)_{r_1} A^s(G)_{r_2} \subset A^s(G)_{r_1+r_2}$ for all $r_1, r_2 \geq 0$;*
- (iii) *$\bigcup_{r \geq 0} A^s(G)_r$ is dense in $A^s(G)$.*

Proof. Note that $a \in A^s(G)_r$ if and only if

- $a \in \mathbb{C}_{L,0}[G; s]$, and
- $l(g) \leq r$ whenever $(gx, g, x) \in \text{supp}(a(t))$ for some $t \geq 0$.

Properties (i) and (iii) follow immediately.

For (ii), if $a \in A^s(G)_{r_1}$, $b \in A^s(G)_{r_2}$, and $(gx, g, x) \in \text{supp}(a(t)b(t))$ for some t , then by Lemma 3.3, $(gx, g, x) = (gx, gh^{-1}, hx)(hx, h, x)$ for some $(gx, gh^{-1}, hx) \in \text{supp}(a(t))$ and $(hx, h, x) \in \text{supp}(b(t))$. Thus $l(g) \leq l(gh^{-1}) + l(h) \leq r_1 + r_2$ so $ab \in A^s(G)_{r_1+r_2}$. \square

Note that if S is an open subset of $\Gamma \ltimes X$, then S generates an open subgroupoid of $\Gamma \ltimes X$ (cf. [7, Lemma 5.2]).

Definition 3.9. *Let G be an open subgroupoid of $\Gamma \ltimes X$ and let $r \geq 0$. The extension of G by r , denoted by G^{+r} , is the open subgroupoid of $\Gamma \ltimes X$ generated by $G \cup \{(gx, g, x) : x \in G^{(0)}, l(g) \leq r\}$.*

Lemma 3.10. *Let G be an open subgroupoid of $\Gamma \ltimes X$ and let $r, s \geq 0$. Then $A^s(G) \cdot A_r^s \cup A_r^s \cdot A^s(G) \subset A^s(G^{+r})$.*

Proof. Follows from Lemma 3.3. \square

Lemma 3.11. *Let G be an open subgroupoid of $\Gamma \ltimes X$, and let $r_1, r_2 \geq 0$. Then $(G^{+r_1})^{+r_2} \subset G^{+(r_1+r_2)}$.*

Proof. It suffices to show that

$$\{(gx, g, x) : x \in (G^{+r_1})^{(0)}, l(g) \leq r_2\} \subset G^{+(r_1+r_2)}.$$

Pick such an element (gx, g, x) . There exists $h \in \Gamma$ with $l(h) \leq r_1$ and $hx \in G^{(0)}$. Thus (gx, gh^{-1}, hx) and (hx, h, x) are in $G^{+(r_1+r_2)}$ so $(gx, g, x) = (gx, gh^{-1}, hx)(hx, h, x) \in G^{+(r_1+r_2)}$. \square

Definition 3.12. [7] *An action $\Gamma \curvearrowright X$ has dynamic asymptotic dimension d if d is the smallest number with the following property: for every open relatively compact subset K of the transformation groupoid $\Gamma \ltimes X$, there are open subsets U_0, \dots, U_d of X covering*

$$\{x \in X : (gx, g, x) \in K \text{ or } (x, g, g^{-1}x) \in K \text{ for some } g \in \Gamma\}$$

such that for each i , the subgroupoid of $\Gamma \ltimes X$ generated by

$$\{(gx, g, x) \in K : x, gx \in U_i\}$$

is relatively compact.

As mentioned in the introduction, many interesting actions have finite dynamic asymptotic dimension. For example, it was shown in [7] that all free minimal \mathbb{Z} -actions on compact spaces (such as irrational rotation of the circle) have dynamic asymptotic dimension one, and that groups with finite asymptotic dimension act with finite dynamic asymptotic dimension on some compact space.

The main consequence of having finite dynamic asymptotic dimension that we will use is the following.

Lemma 3.13. [6, Lemma 5.3] *Suppose that $\Gamma \curvearrowright X$ has dynamic asymptotic dimension d . Then for any $r \geq 0$, there is an open cover $\{U_0, \dots, U_d\}$ of X such that for each $i \in \{0, \dots, d\}$, if G_i is the subgroupoid of $\Gamma \ltimes X$ generated by*

$$\left\{ (gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot U_i, l(g) \leq r \right\},$$

then G_i^{+r} is relatively compact.

We also note that when $1 \leq d < \infty$ and $r \geq 0$, if we set $W_0 = \bigcup_{i=0}^{d-1} U_i$ and $W_1 = U_d$ with $\{U_0, \dots, U_d\}$ as in the lemma, and G_i is the subgroupoid generated by $\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot W_i, l(g) \leq r\}$, then G_i^{+r} is also relatively compact.

4. CONTROLLED K -THEORY

In this section, we recall some definitions and facts from our framework of controlled (or quantitative) K -theory in [2].

Definition 4.1. *A filtered Banach algebra is a Banach algebra A with a family $(A_r)_{r \geq 0}$ of linear subspaces such that*

- $A_{r_1} \subset A_{r_2}$ if $r_1 \leq r_2$;
- $A_{r_1} A_{r_2} \subset A_{r_1+r_2}$ for all $r_1, r_2 \geq 0$;
- $\bigcup_{r \geq 0} A_r$ is dense in A .

If A is unital with unit 1_A , then we require $1_A \in A_r$ for all $r \geq 0$.

We showed in Lemma 3.8 that if G is an open subgroupoid of $\Gamma \ltimes X$ and $s \geq 0$, then $A^s(G)$ is a filtered Banach algebra with filtration

$$A^s(G)_r = \{a \in \mathbb{C}_{L,0}[G; s] : \text{prop}_\Gamma(a(t)) \leq r \text{ for all } t\}.$$

Definition 4.2. *Let A be a filtered Banach algebra. For $0 < \varepsilon < \frac{1}{20}$, $r \geq 0$, and $N \geq 1$,*

- *an element $e \in A$ is called an (ε, r, N) -idempotent if $\|e^2 - e\| < \varepsilon$, $e \in A_r$, and $\max(\|e\|, \|1_{\bar{A}} - e\|) \leq N$.*
- *if A is unital, an element $u \in A$ is called an (ε, r, N) -invertible if $u \in A_r$, $\|u\| \leq N$, and there exists $v \in A_r$ with $\|v\| \leq N$ such that $\max(\|uv - 1\|, \|vu - 1\|) < \varepsilon$.*

We will use the terms quasi-idempotent and quasi-invertible when the precise parameters are not crucial.

Definition 4.3. Let A be a filtered Banach algebra.

- Two (ε, r, N) -idempotents e_0 and e_1 in A are (ε', r', N') -homotopic for some $\varepsilon' \geq \varepsilon$, $r' \geq r$, and $N' \geq N$ if there exists a norm-continuous path $(e_t)_{t \in [0,1]}$ of (ε', r', N') -idempotents in A from e_0 to e_1 . Equivalently, there exists an (ε', r', N') -idempotent $e \in C([0,1], A)$ such that $e(0) = e_0$ and $e(1) = e_1$.
- If A is unital, two (ε, r, N) -invertibles u_0 and u_1 in A are (ε', r', N') -homotopic for some $\varepsilon' \geq \varepsilon$, $r' \geq r$, and $N' \geq N$ if there exists a norm-continuous path $(u_t)_{t \in [0,1]}$ of (ε', r', N') -invertibles in A from u_0 to u_1 . Equivalently, there exists an (ε', r', N') -invertible $u \in C([0,1], A)$ such that $u(0) = u_0$ and $u(1) = u_1$.

Given a filtered L_p operator algebra A , we denote by $\text{Idem}^{\varepsilon, r, N}(A)$ the set of (ε, r, N) -idempotents in A . We set $\text{Idem}_n^{\varepsilon, r, N}(A) = \text{Idem}^{\varepsilon, r, N}(M_n(A))$ for each positive integer n . Then we have inclusions $\text{Idem}_n^{\varepsilon, r, N}(A) \hookrightarrow \text{Idem}_{n+1}^{\varepsilon, r, N}(A)$ given by $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$, and we set

$$\text{Idem}_\infty^{\varepsilon, r, N}(A) = \bigcup_{n \in \mathbb{N}} \text{Idem}_n^{\varepsilon, r, N}(A).$$

Consider the equivalence relation \sim on $\text{Idem}_\infty^{\varepsilon, r, N}(A)$ defined by $e \sim f$ if e and f are $(4\varepsilon, r, 4N)$ -homotopic in $M_\infty(A)$. We will denote the equivalence class of $e \in \text{Idem}_\infty^{\varepsilon, r, N}(A)$ by $[e]$. We will sometimes write $[e]_{\varepsilon, r, N}$ if we wish to emphasize the parameters.

We define addition on $\text{Idem}_\infty^{\varepsilon, r, N}(A)/\sim$ by $[e] + [f] = [\text{diag}(e, f)]$, where $\text{diag}(e, f) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. Then $\text{Idem}_\infty^{\varepsilon, r, N}(A)/\sim$ becomes an abelian semi-group with identity $[0]$. If A is unital, then we define $K_0^{\varepsilon, r, N}(A)$ to be the Grothendieck group of $\text{Idem}_\infty^{\varepsilon, r, N}(A)/\sim$. If A is non-unital, then we define

$$K_0^{\varepsilon, r, N}(A) = \ker(\pi_* : K_0^{\varepsilon, r, N}(A^+) \rightarrow K_0^{\varepsilon, r, N}(\mathbb{C})),$$

where A^+ is the unitization of A and $\pi : A^+ \rightarrow \mathbb{C}$ is the quotient homomorphism.

Given a unital filtered L_p operator algebra A , we denote by $GL^{\varepsilon, r, N}(A)$ the set of (ε, r, N) -invertibles in A . For each positive integer n , we set $GL_n^{\varepsilon, r, N}(A) = GL^{\varepsilon, r, N}(M_n(A))$. Then we have inclusions $GL_n^{\varepsilon, r, N}(A) \hookrightarrow GL_{n+1}^{\varepsilon, r, N}(A)$ given by $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, and we set

$$GL_\infty^{\varepsilon, r, N}(A) = \bigcup_{n \in \mathbb{N}} GL_n^{\varepsilon, r, N}(A).$$

Consider the equivalence relation \sim on $GL_\infty^{\varepsilon, r, N}(A)$ given by $u \sim v$ if u and v are $(4\varepsilon, 2r, 4N)$ -homotopic in $M_\infty(A)$. We will denote the equivalence class of $u \in GL_\infty^{\varepsilon, r, N}(A)$ by $[u]$. We will sometimes write $[u]_{\varepsilon, r, N}$ if we wish to emphasize the parameters.

We define addition on $GL_\infty^{\varepsilon,r,N}(A)/\sim$ by $[u] + [v] = [\text{diag}(u, v)]$. Then $GL_\infty^{\varepsilon,r,N}(A)/\sim$ is an abelian group, which we define to be $K_1^{\varepsilon,r,N}(A)$. If A is non-unital, then we define

$$K_1^{\varepsilon,r,N}(A) = \ker(\pi_* : K_1^{\varepsilon,r,N}(A^+) \rightarrow K_1^{\varepsilon,r,N}(\mathbb{C})),$$

where A^+ is the unitization of A and $\pi : A^+ \rightarrow \mathbb{C}$ is the quotient homomorphism.

There are canonical homomorphisms

$$\iota_*^{\varepsilon,\varepsilon',r,r',N,N'} : K_*^{\varepsilon,r,N}(A) \rightarrow K_*^{\varepsilon',r',N'}(A)$$

for $0 < \varepsilon \leq \varepsilon' < \frac{1}{20}$, $0 \leq r \leq r'$, and $1 \leq N \leq N'$, which we may think of as relaxation of control maps.

If e is an (ε, r, N) -idempotent in a unital filtered Banach algebra A , then we may apply holomorphic functional calculus to get an idempotent $c(e) \in A$. This gives us a group homomorphism

$$c_0 : K_0^{\varepsilon,r,N}(A) \rightarrow K_0(A).$$

Also, every (ε, r, N) -invertible is actually invertible so we have a group homomorphism

$$c_1 : K_1^{\varepsilon,r,N}(A) \rightarrow K_1(A)$$

given by $[u]_{\varepsilon,r,N} \mapsto [u]$. We sometimes refer to these homomorphisms as comparison maps.

Proposition 4.4.

- (i) Let A be a filtered L_p operator algebra. Let f be an idempotent in $M_n(\tilde{A})$, and let $0 < \varepsilon < \frac{1}{20}$. Then there exist $r \geq 0$, $N \geq 1$, and $[e] \in K_0^{\varepsilon,r,N}(A)$ with $e \in \text{Idem}_n^{\varepsilon,r,N}(\tilde{A})$ such that $c_0([e]) = [f]$ in $K_0(A)$.
- (ii) Let A be a filtered L_p operator algebra. Let u be an invertible element in $M_n(\tilde{A})$, and let $0 < \varepsilon < \frac{1}{20}$. Then there exist $r \geq 0$, $N \geq 1$, and $[v] \in K_1^{\varepsilon,r,N}(A)$ with $v \in GL_n^{\varepsilon,r,N}(\tilde{A})$ such that $c_1([v]) = [u]$ in $K_1(A)$.

Proposition 4.5.

- (i) There exists a non-decreasing function $P : [1, \infty) \rightarrow [1, \infty)$ such that for any filtered L_p operator algebra A , if $0 < \varepsilon < \frac{1}{20P(N)}$, and $[e]_{\varepsilon,r,N}, [f]_{\varepsilon,r,N} \in K_0^{\varepsilon,r,N}(A)$ are such that $c_0([e]) = c_0([f])$ in $K_0(A)$, then there exist $r' \geq r$ and $N' \geq N$ such that $[e]_{P(N)\varepsilon,r',N'} = [f]_{P(N)\varepsilon,r',N'}$ in $K_0^{P(N)\varepsilon,r',N'}(A)$.
- (ii) Let A be a filtered L_p operator algebra. Suppose that $0 < \varepsilon < \frac{1}{20}$, and $[u]_{\varepsilon,r,N}, [v]_{\varepsilon,r,N} \in K_1^{\varepsilon,r,N}(A)$ are such that $c_1([u]) = c_1([v])$ in $K_1(A)$. Then there exist $r' \geq r$ and $N' \geq N$ such that $[u]_{\varepsilon,r',N'} = [v]_{\varepsilon,r',N'}$ in $K_1^{\varepsilon,r',N'}(A)$.

The notion of exact sequences is relaxed to allow controlled changes in the parameters.

Definition 4.6. A control pair is a pair (λ, h) , where

- $\lambda : [1, \infty) \rightarrow [1, \infty)$ is a non-decreasing function;
- $h : (0, \frac{1}{20}) \times [1, \infty) \rightarrow [1, \infty)$ is a function such that $h(\cdot, N)$ is non-increasing for fixed N .

We will write λ_N for $\lambda(N)$, and $h_{\varepsilon, N}$ for $h(\varepsilon, N)$.

Given two control pairs (λ, h) and (λ', h') , we write $(\lambda, h) \leq (\lambda', h')$ if $\lambda_N \leq \lambda'_N$ and $h_{\varepsilon, N} \leq h'_{\varepsilon, N}$ for all $\varepsilon \in (0, \frac{1}{20})$ and $N \geq 1$.

Given a filtered L_p operator algebra A , we consider the families

$$\mathcal{K}_0(A) = (K_0^{\varepsilon, r, N}(A))_{0 < \varepsilon < \frac{1}{20}, r \geq 0, N \geq 1}$$

and

$$\mathcal{K}_1(A) = (K_1^{\varepsilon, r, N}(A))_{0 < \varepsilon < \frac{1}{20}, r \geq 0, N \geq 1}.$$

Definition 4.7. Let A and B be filtered L_p operator algebras, and let (λ, h) be a control pair. A (λ, h) -controlled morphism $\mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$, where $i, j \in \{0, 1\}$, is a family

$$\mathcal{F} = (F^{\varepsilon, r, N})_{0 < \varepsilon < \frac{1}{20\lambda_N}, r \geq 0, N \geq 1}$$

of group homomorphisms

$$F^{\varepsilon, r, N} : K_i^{\varepsilon, r, N}(A) \rightarrow K_j^{\lambda_N \varepsilon, h_{\varepsilon, N} r, \lambda_N}(B)$$

such that whenever $0 < \varepsilon \leq \varepsilon' < \frac{1}{20\lambda_{N'}}$, $h_{\varepsilon, N} r \leq h_{\varepsilon', N'} r'$, and $N \leq N'$, we have the following commutative diagram:

$$\begin{array}{ccc} K_i^{\varepsilon, r, N}(A) & \xrightarrow{\iota_i} & K_i^{\varepsilon', r', N'}(A) \\ F^{\varepsilon, r, N} \downarrow & & \downarrow F^{\varepsilon', r', N'} \\ K_j^{\lambda_N \varepsilon, h_{\varepsilon, N} r, \lambda_N}(B) & \xrightarrow{\iota_j} & K_j^{\lambda_{N'} \varepsilon', h_{\varepsilon', N'} r', \lambda_{N'}}(B) \end{array}$$

where ι_i and ι_j are the canonical inclusions. We say that \mathcal{F} is a controlled morphism if it is a (λ, h) -controlled morphism for some control pair (λ, h) .

Definition 4.8. Let A, B , and C be filtered L_p operator algebras, and let (λ, h) be a control pair. Let $\mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$ be a $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$ -controlled morphism, and let $\mathcal{G} : \mathcal{K}_j(B) \rightarrow \mathcal{K}_l(C)$ be a $(\lambda^{\mathcal{G}}, h^{\mathcal{G}})$ -controlled morphism, where $i, j, l \in \{0, 1\}$. Then the composition $\mathcal{K}_i(A) \xrightarrow{\mathcal{F}} \mathcal{K}_j(B) \xrightarrow{\mathcal{G}} \mathcal{K}_l(C)$ is said to be (λ, h) -exact (at $\mathcal{K}_j(B)$) if

- $G^{\lambda_N^{\mathcal{F}} \varepsilon, h_{\varepsilon, N}^{\mathcal{F}} r, \lambda_N^{\mathcal{F}}} \circ F^{\varepsilon, r, N} = 0$ for all ε, r, N ;

- for any $0 < \varepsilon < \frac{1}{20 \max((\lambda^{\mathcal{F}} \cdot \lambda)_N, \lambda_N^{\mathcal{G}})}$, $r \geq 0$, $N \geq 1$, and $y \in K_j^{\varepsilon, r, N}(B)$ such that $G^{\varepsilon, r, N}(y) = 0$ in $K_l^{\lambda_N^{\mathcal{G}}, h_{\varepsilon, N}^{\mathcal{G}}, \lambda_N^{\mathcal{G}}}(C)$, there exists $x \in K_i^{\lambda_N^{\mathcal{G}}, h_{\varepsilon, N}^{\mathcal{G}}, \lambda_N^{\mathcal{G}}}(A)$ such that $F^{\lambda_N^{\mathcal{G}}, h_{\varepsilon, N}^{\mathcal{G}}, \lambda_N^{\mathcal{G}}}(x) = \iota_j(y)$ in $K_j^{(\lambda^{\mathcal{F}} \cdot \lambda)_N, (h^{\mathcal{F}} \cdot h)_{\varepsilon, N}, (\lambda^{\mathcal{F}} \cdot \lambda)_N}(B)$.

A sequence of controlled morphisms

$$\cdots \rightarrow \mathcal{K}_{i_{k-1}}(A_{k-1}) \rightarrow \mathcal{K}_{i_k}(A_k) \rightarrow \mathcal{K}_{i_{k+1}}(A_{k+1}) \rightarrow \mathcal{K}_{i_{k+2}}(A_{k+2}) \rightarrow \cdots$$

is (λ, h) -exact if the composition $\mathcal{K}_{i_{k-1}}(A_{k-1}) \rightarrow \mathcal{K}_{i_k}(A_k) \rightarrow \mathcal{K}_{i_{k+1}}(A_{k+1})$ is (λ, h) -exact for every k .

5. MAIN THEOREM

In this section, we shall prove the main theorem of this paper, Theorem 1.3, which we restate here.

Theorem 5.1. *Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X . Then the L_p Baum-Connes conjecture holds for Γ with coefficients in $C(X)$ for $p \in (1, \infty)$.*

Recall that the L_p reduced crossed product is defined as follows:

Let A be a closed subalgebra of $B(L_p(Z, \mu))$ for some measure space (Z, μ) and $p \in (1, \infty)$. Let Γ be a countable discrete group acting on A by isometric automorphisms. Define $A\Gamma = \{\sum_{g \in \Gamma} a_g g : a_g \in A\}$ with the product given by

$$\left(\sum_{g \in \Gamma} a_g g \right) \left(\sum_{h \in \Gamma} b_h h \right) = \sum_{g, h \in \Gamma} a_g \alpha_g(b_h) gh,$$

where α denotes the Γ -action on A . There is a natural faithful representation of $A\Gamma$ on $\ell_p(\Gamma, L_p(Z, \mu))$ given by

$$\begin{aligned} (a\xi)(h) &= \alpha_{g^{-1}}(a)\xi(h), \\ (g\xi)(h) &= \xi(g^{-1}h) \end{aligned}$$

for $a \in A$, $g, h \in \Gamma$, and $\xi \in \ell_p(\Gamma, L_p(Z, \mu))$. We then define the L_p reduced crossed product $A \rtimes_{\lambda, p} \Gamma$ to be the operator norm closure of $A\Gamma$ in $B(\ell_p(\Gamma, L_p(Z, \mu)))$.

We formulate the L_p Baum-Connes conjecture with coefficients by considering the L_p reduced crossed product instead of the reduced crossed product C^* -algebra on the right-hand side of the assembly map.

Conjecture 5.2 (L_p Baum-Connes conjecture for Γ with coefficients in A). *The homomorphism*

$$\mu_p : K_*^{\Gamma}(\underline{E}\Gamma; A) \rightarrow K_*(A \rtimes_{\lambda, p} \Gamma)$$

is an isomorphism, where the left-hand side is the equivariant K -homology of the classifying space for proper Γ -actions with coefficients in A , and the right-hand side is the K -theory of the L_p reduced crossed product.

We will use a particular model for $\underline{E}\Gamma$, namely $\bigcup_{s \geq 0} P_s(\Gamma)$ equipped with the ℓ_1 metric [1, Section 2], where $P_s(\Gamma)$ is the Rips complex of Γ at scale s . The Baum-Connes assembly map can then be thought of as a map

$$\mu_p : \lim_{s \rightarrow \infty} K_*^\Gamma(P_s(\Gamma); A) \rightarrow K_*(A \rtimes_{\lambda, p} \Gamma).$$

Just as in the C^* -algebra setting (see for example [6, Theorem A.3]), and with essentially the same proof, we have a commutative diagram

$$\begin{array}{ccc} K_*^\Gamma(P_s(\Gamma); A) & \xrightarrow{\mu_p} & K_*(A \rtimes_{\lambda, p} \Gamma) \\ \downarrow & & \downarrow \\ K_*(C_L^{*,p}(P_s(\Gamma); A)) & \xrightarrow{\epsilon_0} & K_*(C^{*,p}(P_s(\Gamma); A)) \end{array}$$

where the vertical maps are isomorphisms, which allows us to identify the L_p Baum-Connes assembly map μ_p with the evaluation-at-zero map

$$\epsilon_0 : \lim_{s \rightarrow \infty} K_*(C_L^{*,p}(P_s(\Gamma); A)) \rightarrow \lim_{s \rightarrow \infty} K_*(C^{*,p}(P_s(\Gamma); A)).$$

In the case where $A = C(X)$, this is the assembly map that we defined in Section 2.

Thus proving the L_p Baum-Connes conjecture for Γ with coefficients in $C(X)$ amounts to proving that the evaluation-at-zero map ϵ_0 induces an isomorphism, which is equivalent to proving that

$$\lim_{s \rightarrow \infty} K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)) = 0.$$

The proof that we present is modeled after the proof in the C^* -algebraic setting in [6], and the idea is as follows:

We want to show that for any $s_0 \geq 0$ and any $x \in K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s_0))$, there exists $s \geq s_0$ such that the map

$$K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s_0)) \rightarrow K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s))$$

induced by inclusion sends x to 0. By Proposition 4.4, we may pass over to the controlled setting, so it suffices to show that the corresponding homomorphism between the controlled K -theory groups is zero. Via an induction argument using a controlled exact Mayer-Vietoris sequence, it comes down to showing that if G is an open subgroupoid of $\Gamma \rtimes X$ such that $G \subset \{(gx, g, x) \in \Gamma \rtimes X : l(g) \leq s\}$ for some $s \geq 0$, then $K_*(C_{L,0}^{*,p}(G; s)) = 0$. For that, roughly speaking, the point is that the assumption on G essentially makes the associated Rips complex contractible so homotopy invariance will give us the result.

5.1. Base case. Recall that we use the shorthand $A^s(G)$ for $C_{L,0}^{*,p}(G; s)$. As mentioned above, the base case of our induction argument involves proving the following.

Proposition 5.3. *Let G be an open subgroupoid of $\Gamma \rtimes X$ such that $G \subset \{(gx, g, x) \in \Gamma \rtimes X : l(g) \leq s\}$ for some $s \geq 0$. Then $K_*(A^s(G)) = 0$.*

Before getting into the proof of the proposition, we need to fix some terminology that is standard in the C^* -algebraic setting but perhaps less so when Hilbert spaces are replaced by other Banach spaces.

Definition 5.4. Let E be a complex Banach space. We say that $T \in B(E)$ is a partial isometry if $\|T\| \leq 1$ and there exists $S \in B(E)$ such that $\|S\| \leq 1$ and ST, TS are idempotents.

Remark 5.5. If (Z, μ) is a σ -finite measure space, $p \in [1, \infty) \setminus \{2\}$, and $T \in B(L_p(Z, \mu))$ is an isometric (but not necessarily surjective) linear map, then it follows from Lamperti's theorem [9] that T is a partial isometry in the sense above, and one can find S as in the definition such that $ST = I$ (also see [14, Section 6]). Hereafter, we will denote such an S by $T^{(*)}$.

Definition 5.6. If $A \subset B(L_p(\mu))$ is an L_p operator algebra, then we say that $b \in B(L_p(\mu))$ is a multiplier of A if $bA \subset A$ and $Ab \subset A$. We say that $b \in B(L_p(\mu))$ is an isometric multiplier of A if b is an isometry and both b and $b^{(*)}$ are multipliers of A . The set of all multipliers of A is denoted by $M(A)$.

One can check that $M(A)$ is also an L_p operator algebra.

Fixing G as above and $s \geq 0$, we make the following definitions:

- $P_s(G) = \{(z, x) \in P_s(\Gamma) \times X : (gx, g, x) \in G \text{ for all } g \in \text{supp}(z)\}$.
- $Z_G = (Z_s \times X) \cap P_s(G)$.
- $E_G = \ell_p(Z_G, \ell_p \otimes \ell_p(\Gamma)) = \ell_p(Z_G) \otimes \ell_p(\Gamma)$.

Note that E_G is a subspace of E_s . Moreover, the faithful representation of $C^{*,p}(G; s)$ on E_s restricts to a faithful representation on E_G . Thus we will consider $C^{*,p}(G; s)$ as faithfully represented on E_G , and $A^s(G) := C_{L,0}^{*,p}(G; s)$ as faithfully represented on $L_p([0, \infty), E_G)$.

Also, if $(z, x) \in P_s(G)$ and $\text{supp}(z) = \{g_1, \dots, g_n\}$, then $\{e, g_1, \dots, g_n\}$ also spans a simplex Δ in $P_s(\Gamma)$ such that $\Delta \times \{x\}$ is contained in $P_s(G)$. Hence the family of functions

$$H_r : P_s(G) \rightarrow P_s(G), (z, x) \mapsto ((1-r)z + re, x) \quad (0 \leq r \leq 1)$$

defines a homotopy between the identity map on $P_s(G)$ and the projection onto the subset $\{(z, x) \in P_s(G) : z = e\}$, which we may identify with the unit space $G^{(0)}$.

In the definition of $\mathbb{C}[\Gamma \curvearrowright X; s]$, we may use $\mathcal{K}_\Gamma^\infty$, the algebra of compact operators on $(\bigoplus_{n=0}^\infty \ell_p \otimes \ell_p(\Gamma))_p \cong (\bigoplus_{n=0}^\infty \ell_p)_p \otimes \ell_p(\Gamma)$, thereby obtaining another L_p Roe algebra $C^{*,p}(\Gamma \curvearrowright X; \mathcal{K}_\Gamma^\infty; s)$. Moreover, fixing an isometric isomorphism $\phi : \ell_p \xrightarrow{\cong} (\bigoplus_{n=0}^\infty \ell_p)_p$ gives an isomorphism

$$C^{*,p}(\Gamma \curvearrowright X; s) \cong C^{*,p}(\Gamma \curvearrowright X; \mathcal{K}_\Gamma^\infty; s).$$

We also have the corresponding statements for the L_p localization algebras and obstruction algebras defined earlier.

For each n , define an isometry $u_{n,0} : \ell_p \rightarrow (\bigoplus_{n=0}^\infty \ell_p)_p$ by inclusion as the n th summand, and define $u_{n,0}^{(*)} : (\bigoplus_{n=0}^\infty \ell_p)_p \rightarrow \ell_p$ by projection onto

the n th summand. Then $u_{n,0}^{(*)}u_{n,0} = I_{\ell_p}$ for all n , and $u_{n,0}^{(*)}u_{m,0} = 0$ when $n \neq m$. Define $u_n : L_p([0, \infty), E_G) \rightarrow L_p([0, \infty), E_G^\infty)$ to be the operator induced by tensoring $u_{n,0}$ with the identity on the other factors, where $E_G^\infty = (\bigoplus_{n=0}^\infty E_G)_p$, and define $u_n^{(*)}$ similarly using $u_{n,0}^{(*)}$. Then $u_n^{(*)}u_n = I$ for all n , and $u_n^{(*)}u_m = 0$ when $n \neq m$.

Given $a \in B(\ell_p)$, consider $a^\infty = a \oplus a \oplus \cdots \in B((\bigoplus_{n=0}^\infty \ell_p)_p)$. Then $\mu(a) = a^\infty$ is an isometric homomorphism $B(\ell_p) \rightarrow B((\bigoplus_{n=0}^\infty \ell_p)_p)$. We may also consider the isometry $v \in B((\bigoplus_{n=0}^\infty \ell_p)_p)$ given by the right shift taking the n th summand onto the $(n+1)$ st summand. Denote by $v^{(*)} \in B((\bigoplus_{n=0}^\infty \ell_p)_p)$ the left shift. Then $v\mu(a)v^{(*)} = 0 \oplus a \oplus a \oplus \cdots$ for all $a \in B(\ell_p)$. With $u_{0,0}$ as above, $u_{0,0}au_{0,0}^{(*)}$ is given by $a \oplus 0 \oplus 0 \oplus \cdots$ for all $a \in B(\ell_p)$. Now $a \mapsto \mu^{+1}(a) := v\mu(a)v^{(*)}$ and $a \mapsto \mu^0(a) := u_{0,0}au_{0,0}^{(*)}$ are bounded homomorphisms $B(\ell_p) \rightarrow B((\bigoplus_{n=0}^\infty \ell_p)_p)$.

Now given $a \in B(L_p([0, \infty), E_G))$, consider the bounded linear operator $\mu(a) = a^\infty$ on $L_p([0, \infty), E_G^\infty)$. Proceeding similarly as above, we get bounded homomorphisms

$$\mu, \mu^{+1}, \mu^0 : B(L_p([0, \infty), E_G)) \rightarrow B(L_p([0, \infty), E_G^\infty)).$$

Moreover, each of them maps $A^s(G)$ into $A^s(G; \mathcal{K}_\Gamma^\infty)$, and $M(A^s(G))$ into $M(A^s(G; \mathcal{K}_\Gamma^\infty))$.

The following lemma (whose proof we omit) is the L_p version of a fairly standard result in the K -theory of C^* -algebras.

Lemma 5.7. *Let $\alpha : A \rightarrow C$ be a bounded homomorphism of L_p operator algebras with $C \subset B(L_p(\mu))$, and let $v \in B(L_p(\mu))$ be an isometric multiplier of C . Then the map $a \mapsto v\alpha(a)v^{(*)}$ is a bounded homomorphism from A to C , and induces the same map as α in K -theory.*

More generally, if $v \in B(L_p(\mu))$ is a partial isometry and a multiplier of C such that $v^{()}$ is also a multiplier of C and $\alpha(a)v^{(*)}v = \alpha(a)$ for all $a \in A$, then the map $a \mapsto v\alpha(a)v^{(*)}$ is a bounded homomorphism from A to C , and induces the same map as α in K -theory.*

Lemma 5.8. $K_*(M(A^s(G))) = 0$.

Proof. Note that $\mu, \mu^{+1} : M(A^s(G)) \rightarrow M(A^s(G; \mathcal{K}_\Gamma^\infty))$ induce the same map in K -theory by the previous lemma. Moreover, since $\mu^0(a)\mu^{+1}(a) = \mu^{+1}(a)\mu^0(a) = 0$ for all $a \in M(A^s(G))$ and $\mu = \mu^0 + \mu^{+1}$, the induced maps in K -theory satisfy $\mu_* = \mu_*^0 + \mu_*^{+1} = \mu_*^0 + \mu_*$. Hence $\mu_*^0 = 0$. But μ^0 induces an isomorphism in K -theory so $K_*(M(A^s(G))) = 0$. \square

For each $z \in Z_s$ such that $(z, x) \in P_s(G)$ for some $x \in X$, let E_z be a copy of ℓ_p so that we have an isometric isomorphism $\ell_p \cong (\bigoplus_{z \in Z} E_z)_p$, and let $w_z : \ell_p \otimes \ell_p(\Gamma) \rightarrow \ell_p \otimes \ell_p(\Gamma)$ be an isometry with range $E_z \otimes \ell_p(\Gamma)$. For each $r \in \mathbb{Q} \cap [0, 1]$, define $w(r) : \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma) \rightarrow \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$ by $\delta_{z,x} \otimes \eta \mapsto \delta_{(1-r)z+re,x} \otimes w_z\eta$, which is a well-defined isometry.

For $t \geq 0$ and $n \in \mathbb{N} \cup \{\infty\}$, define an isometry

$$v_n(t) : \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma) \rightarrow \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$$

in the following way:

- (i) for $t < n$, $v_n(t) = w(0)$ (in particular, $v_\infty(t) = w(0)$ for all t);
- (ii) for $m \in [n, 2n) \cap \mathbb{N}$, $t \in [m, m+1)$, $(z, x) \in Z_G$ with $z \neq e$, and $\eta \in \ell_p \otimes \ell_p(\Gamma)$,

$$v_n(t)(\delta_{z,x} \otimes \eta) = \left(\left| \cos \left(\frac{\pi}{2}(t-m) \right) \right| w \left(\frac{m-n}{n} \right) + \left| \sin \left(\frac{\pi}{2}(t-m) \right) \right| w \left(\frac{m+1-n}{n} \right) \right) (\delta_{z,x} \otimes \eta);$$

- (iii) for $t \in [n, 2n)$, $(e, x) \in Z_G$, and $\eta \in \ell_p \otimes \ell_p(\Gamma)$,

$$v_n(t)(\delta_{e,x} \otimes \eta) = \delta_{e,x} \otimes w_e \eta;$$

- (iv) for $t \geq 2n$, $v_n(t) = w(1)$.

One can check that the map $[0, \infty) \rightarrow B(\ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)), t \mapsto v_n(t)$, is norm continuous for each n . Now define an isometry

$$v_n : L_p([0, \infty), E_G) \rightarrow L_p([0, \infty), E_G)$$

for each n by $(v_n \xi)(t) = v_n(t)(\xi(t))$ for $\xi \in L_p([0, \infty), E_G)$.

Let $a \in \mathbb{C}_{L,0}[G; s]$ and let $T = a(t)$ for some fixed $t \in [0, \infty)$. The matrix entries $(v_n(t)Tv_n(t)^{(*)})_{y,z}(x)$ of $v_n(t)Tv_n(t)^{(*)}$ will be linear combinations of at most four terms of the form $w_z^{(*)}T_{H_{r_1}(y), H_{r_2}(z)}(x)w_y^{(*)}$, where $r_1, r_2 \in \mathbb{Q} \cap [0, 1]$ and $|r_1 - r_2| < \frac{1}{m}$ whenever $t > \frac{2}{m}$. It follows that the Rips-propagation of $v_n(t)a(t)v_n(t)^{(*)}$ is at most $\text{prop}_{\text{Rips}}a(t) + \min(1, \frac{2}{|t-1|})$, so $v_nav_n^{(*)} \in \mathbb{C}_{L,0}[G; s]$.

Also, the operators $S_t := v_{n+1}(t)v_n(t)^{(*)}$ on $\ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$ have matrix entries $(S_t)_{y,z}$ that act as constant functions $X \rightarrow B(\ell_p \otimes \ell_p(\Gamma))$, their Rips-propagation tends to zero as $t \rightarrow \infty$, and they have Γ -propagation at most s for all t . Hence $v_{n+1}v_n^{(*)}$ is a multiplier of $A^s(G)$ for all n .

Lemma 5.9. *Let A be a unital Banach algebra, and let I be an ideal in A . Define the double of A along I to be $D = \{(a, b) \in A : a - b \in I\}$. Assume that A has trivial K -theory. Then the inclusion $\iota : I \rightarrow D$ given by $a \mapsto (a, 0)$ induces an isomorphism in K -theory, and the diagonal inclusion $\delta : I \rightarrow D$ given by $a \mapsto (a, a)$ induces the zero map on K -theory.*

Proof. Note that $\iota(I)$ is an ideal in D , and $D/\iota(I)$ is isomorphic to A via the second coordinate projection. Since $K_*(A) = 0$, it follows from the six-term exact sequence that ι induces an isomorphism in K -theory.

On the other hand, δ factors through the diagonal inclusion $A \rightarrow D, a \mapsto (a, a)$, so δ induces the zero map on K -theory since $K_*(A) = 0$. \square

We shall apply the lemma in the case where $A = M(A^s(G))$ and $I = A^s(G)$ to prove the next proposition.

Proposition 5.10. *Let $v_n : L_p([0, \infty), E_G) \rightarrow L_p([0, \infty), E_G)$ be as defined above. Then the maps $a \mapsto v_0 av_0^{(*)}$ and $a \mapsto v_\infty av_\infty^{(*)}$ induce the same map $K_*(A^s(G)) \rightarrow K_*(A^s(G))$.*

Proof. Given $a \in A^s(G)$, define

$$\alpha(a) = \left(\bigoplus_{n=0}^{\infty} v_n av_n^{(*)}, \bigoplus_{n=0}^{\infty} v_\infty av_\infty^{(*)} \right) \in A^s(G; \mathcal{K}_\Gamma^\infty) \oplus A^s(G; \mathcal{K}_\Gamma^\infty).$$

Also define

$$\beta(a) = \left(\bigoplus_{n=0}^{\infty} v_{n+1} av_{n+1}^{(*)}, \bigoplus_{n=0}^{\infty} v_\infty av_\infty^{(*)} \right) \in A^s(G; \mathcal{K}_\Gamma^\infty) \oplus A^s(G; \mathcal{K}_\Gamma^\infty).$$

Let D be the double of $M(A^s(G; \mathcal{K}_\Gamma^\infty))$ along $A^s(G; \mathcal{K}_\Gamma^\infty)$, and let

$$C = \{(c, d) \in D : d = \bigoplus_{n=0}^{\infty} v_\infty av_\infty^{(*)} \text{ for some } a \in A^s(G)\},$$

which is a closed subalgebra of D . Moreover, α and β are bounded homomorphisms with image in C . Consider $w = (w_1, w_2)$, where $w_1 = \bigoplus_{n=0}^{\infty} v_{n+1} v_n^{(*)}$ and $w_2 = \bigoplus_{n=0}^{\infty} v_\infty av_\infty^{(*)}$. Note that $w_1 \in M(A^s(G; \mathcal{K}_\Gamma^\infty))$. We claim that w is a multiplier of C . Indeed, if $(c, d) \in C$, then $w_2 d = d w_2 = d$ so it suffices to show that $c w_1 - d$ and $w_1 c - d$ are in $A^s(G; \mathcal{K}_\Gamma^\infty)$. We will only consider $w_1 c - d$ since the other case is similar. Now $w_1 c - d = w_1(c - d) + (w_1 d - d)$ so it suffices to show that $w_1 d - d \in A^s(G; \mathcal{K}_\Gamma^\infty)$. But $w_1 d - d = (w_1 - w_2)d = \bigoplus_{n=0}^{\infty} (v_{n+1} v_n^{(*)} - v_\infty v_\infty^{(*)}) v_\infty av_\infty^{(*)} \in A^s(G; \mathcal{K}_\Gamma^\infty)$ since $v_n(t) = v_\infty(t)$ for each fixed t and all $n > t$. Similarly, $w^{(*)} = (w_1^{(*)}, w_2^{(*)})$ is a multiplier of C . Now $\beta(a) = w \alpha(a) w^{(*)}$ for all $a \in A$. Moreover, $\alpha(a) w^{(*)} w = \alpha(a)$ so α and β induce the same map $K_*(A^s(G)) \rightarrow K_*(C)$, and thus the same map $K_*(A^s(G)) \rightarrow K_*(D)$ upon composing with the map induced by the inclusion of C into D .

Let u be the isometric multiplier of $A^s(G; \mathcal{K}_\Gamma^\infty)$ induced by the right shift. Then (u, u) is a multiplier of D , and conjugating $\beta(a)$ by (u, u) gives

$$\gamma(a) = \left(0 \oplus \bigoplus_{n=1}^{\infty} v_n av_n^{(*)}, 0 \oplus \bigoplus_{n=1}^{\infty} v_\infty av_\infty^{(*)} \right).$$

Thus β and γ induce the same map $K_*(A) \rightarrow K_*(D)$. On the other hand, the homomorphism $\delta : A^s(G) \rightarrow D$ given by

$$a \mapsto (v_\infty av_\infty^{(*)} \oplus 0 \oplus 0 \oplus \cdots, v_\infty av_\infty^{(*)} \oplus 0 \oplus 0 \oplus \cdots)$$

induces the zero map on K -theory by the previous lemma. Also, $\gamma(a)\delta(a) = \delta(a)\gamma(a) = 0$. Hence

$$\alpha_* = \beta_* = \gamma_* = \gamma_* + \delta_* = (\gamma + \delta)_* : K_*(A^s(G)) \rightarrow K_*(D).$$

Let $\psi_0, \psi_\infty : A^s(G) \rightarrow D$ be the homomorphisms defined by

$$\begin{aligned}\psi_0(a) &= (v_0 a v_0^{(*)} \oplus 0 \oplus 0 \oplus \cdots, 0), \\ \psi_\infty(a) &= (v_\infty a v_\infty^{(*)} \oplus 0 \oplus 0 \oplus \cdots, 0).\end{aligned}$$

Also define $\zeta : A^s(G) \rightarrow D$ by

$$\zeta(a) = \left(0 \oplus \bigoplus_{n=1}^{\infty} v_n a v_n^{(*)}, \bigoplus_{n=0}^{\infty} v_\infty a v_\infty^{(*)} \right).$$

Note that $\zeta(a)\psi_0(a) = \psi_0(a)\zeta(a) = \zeta(a)\psi_\infty(a) = \psi_\infty(a)\zeta(a) = 0$ for all $a \in A^s(G)$. Also, $\psi_0 + \zeta = \alpha$ and $\psi_\infty + \zeta = \gamma + \delta$. Hence

$$(\psi_0)_* + \zeta_* = \alpha_* = (\gamma + \delta)_* = (\psi_\infty)_* + \zeta_*,$$

so ψ_0 and ψ_∞ induce the same maps on K -theory.

Finally, if $\iota : A^s(G) \rightarrow D$ is the inclusion into the first factor (where D is now regarded as the double of $M(A^s(G))$ along $A^s(G)$), then $\psi_i(a)$ is given by the composition

$$a \mapsto v_i a v_i^{(*)} \xrightarrow{\iota} (v_i a v_i^{(*)}, 0) \mapsto (v_i a v_i^{(*)} \oplus 0 \oplus 0 \oplus \cdots, 0),$$

and the last two maps induce isomorphisms in K -theory, so $a \mapsto v_0 a v_0^{(*)}$ and $a \mapsto v_\infty a v_\infty^{(*)}$ induce the same map in K -theory. \square

Now it remains to show that $a \mapsto v_\infty a v_\infty^{(*)}$ induces the identity map on $K_*(A^s(G))$ while $a \mapsto v_0 a v_0^{(*)}$ induces the zero map on $K_*(A^s(G))$. This will complete the proof of Proposition 5.3.

Lemma 5.11. *The map $\phi_\infty : K_*(A^s(G)) \rightarrow K_*(A^s(G))$ induced by conjugation by v_∞ is the identity map.*

Proof. Since $v_\infty(t) = w(0)$ for all t , and $w(0)$ is an isometric multiplier of $A^s(G)$, ϕ_∞ induces the identity map in K -theory. \square

Lemma 5.12. *The map $\phi_0 : K_*(A^s(G)) \rightarrow K_*(A^s(G))$ induced by conjugation by v_0 is the zero map.*

Proof. Let $G^{(0)}$ be the unit space of G , which is an open subgroupoid of $\Gamma \ltimes X$. We may then consider $A^s(G^{(0)})$. In fact, ϕ_0 factors through $K_*(A^s(G^{(0)}))$, i.e., we have a commutative diagram

$$\begin{array}{ccc} K_*(A^s(G)) & \xrightarrow{\phi_0} & K_*(A^s(G)) \\ & \searrow & \nearrow \\ & K_*(A^s(G^{(0)})) & \end{array}$$

so it suffices to show that $K_*(A^s(G^{(0)})) = 0$.

For each $n \in \mathbb{N}$ and $a \in A^s(G^{(0)})$, define

$$a^{(n)}(t) = \begin{cases} a(t-n) & \text{for } t \geq n \\ 0 & \text{for } t < n \end{cases}.$$

Note that $a^{(n)} \in A^s(G^{(0)})$. Now define $\alpha : A^s(G^{(0)}) \rightarrow A^s(G^{(0)}; \mathcal{K}_\Gamma^\infty)$ by $a \mapsto \bigoplus_{n=0}^\infty a^{(n)}$. We also have the “top corner inclusion” $\iota : A^s(G^{(0)}) \rightarrow A^s(G^{(0)}; \mathcal{K}_\Gamma^\infty)$ given by $a \mapsto a \oplus 0 \oplus 0 \oplus \dots$. Using uniform continuity of elements in $A^s(G^{(0)})$ together with arguments similar to those above, we see that $\alpha_* + \iota_* = \alpha_*$ so $\iota_* = 0$. But ι induces an isomorphism in K -theory so it follows that $K_*(A^s(G^{(0)})) = 0$. \square

5.2. Inductive step. Given two open subgroupoids of $\Gamma \ltimes X$, we will consider associated subalgebras of $A^s := C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$. However, the filtrations we put on these subalgebras are not the induced filtrations from A^s .

Definition 5.13. Fix $s_0 \geq 0$. Let G_0 and G_1 be open subgroupoids of $\Gamma \ltimes X$, and let $s \geq s_0$. For $r \geq 0$, define

$$A_r = A^{s_0}(G_0^{+r})_r + A^{s_0}(G_1^{+r})_r + A^s(G_0^{+r} \cap G_1^{+r}),$$

and

$$A = \overline{\bigcup_{r \geq 0} A_r},$$

taking closure in the norm of A^s .

Also define

$$I_r = A^{s_0}(G_0^{+r})_r + A^s(G_0^{+r} \cap G_1^{+r}), \quad J_r = A^{s_0}(G_1^{+r})_r + A^s(G_0^{+r} \cap G_1^{+r}),$$

and

$$I = \overline{\bigcup_{r \geq 0} I_r}, \quad J = \overline{\bigcup_{r \geq 0} J_r}.$$

Lemma 5.14. With notation as above, $(A_r)_{r \geq 0}$ is a filtration for A . Moreover, I and J are ideals in A .

Proof. It is clear that $A_{r_0} \subset A_r$ if $r_0 \leq r$, and that $\bigcup_{r \geq 0} A_r$ is dense in A . By Lemmas 3.3 and 3.10, it follows that for $r_1, r_2 \geq 0$,

$$A^{s_0}(G_i^{+r_1})_{r_1} \cdot A^{s_0}(G_i^{+r_2})_{r_2} \subset A^{s_0}(G_i^{+(r_1+r_2)})_{r_1+r_2}$$

for $i = 0, 1$, while

$$A^{s_0}(G_0^{+r_1})_{r_1} \cdot A^{s_0}(G_1^{+r_2})_{r_2} \subset A^{s_0}(G_0^{+(r_1+r_2)})_{r_1+r_2} \cap A^{s_0}(G_1^{+(r_1+r_2)})_{r_1+r_2}.$$

Also,

$$A^s(G_0^{+r_1} \cap G_1^{+r_1}) \cdot A^s(G_0^{+r_2} \cap G_1^{+r_2}) \subset A^s(G_0^{+(r_1+r_2)} \cap G_1^{+(r_1+r_2)})$$

and

$$\begin{aligned}
A^{s_0}(G_i^{+r_1})_{r_1} \cdot A^s(G_0^{+r_2} \cap G_1^{+r_2}) &\subset A^s((G_0^{+r_2} \cap G_1^{+r_2})^{+r_1}) \\
&\subset A^s((G_0^{+r_2})^{+r_1} \cap (G_1^{+r_2})^{+r_1}) \\
&\subset A^s(G_0^{+(r_1+r_2)} \cap G_1^{+(r_1+r_2)}).
\end{aligned}$$

Hence $(A_r)_{r \geq 0}$ is a filtration for A , while I and J are ideals in A . \square

In [2], we showed the existence of a controlled Mayer-Vietoris sequence under certain hypotheses. Here, we shall state the hypotheses in a slightly less general manner (by omitting certain parameters) that suffices for our application. On the other hand, we also give ourselves a bit more flexibility in terms of propagation control. One can check that the proofs in [2] carry over after adjusting the propagation parameter.

Definition 5.15. Let A be a filtered L_p algebra with filtration $(A_r)_{r \geq 0}$. A controlled Mayer-Vietoris pair for A is a pair $(A_{\Delta_1}, A_{\Delta_2})$ of Banach subalgebras of A associated with a pair (Δ_1, Δ_2) of closed linear subspaces of A_s satisfying the following conditions:

- There exists $\rho : [0, \infty) \rightarrow [0, \infty)$ with $\rho(r) \geq r$ such that for any $r \geq 0$, any positive integer n , and any $x \in M_n(A_r)$, there exist $x_1 \in M_n(\Delta_1 \cap A_{\rho(r)})$ and $x_2 \in M_n(\Delta_2 \cap A_{\rho(r)})$ such that $x = x_1 + x_2$ and $\max(\|x_1\|, \|x_2\|) \leq \|x\|$;
- A_{Δ_i} has filtration $(A_{\Delta_i} \cap A_r)_{r \geq 0}$, and A_{Δ_i} contains $\Delta_i + A_{4s}\Delta_i + \Delta_i A_{4s} + A_{4s}\Delta_i A_{4s}$;
- For any $r \geq 0$, any $\varepsilon > 0$, any positive integer n , any $x \in M_n(A_{\Delta_1, r})$ and $y \in M_n(A_{\Delta_2, r})$ with $\|x - y\| < \varepsilon$, there exists $z \in M_n(A_{\Delta_1, \rho(r)} \cap A_{\Delta_2, \rho(r)})$ such that $\max(\|z - x\|, \|z - y\|) < \varepsilon$, where ρ is as above.

Remark 5.16. If A_{Δ_i} is a closed ideal in A , and we let $\Delta_i = A_{\Delta_i}$, then the second part of the second condition above is automatically satisfied.

Theorem 5.17. [2, Theorem 5.14] There exists a control pair (λ, h) such that for any filtered L_p algebra A and any controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A , we have the following (λ, h) -exact sequences:

$$\begin{array}{ccccccc}
\mathcal{K}_1(A_{\Delta_1} \cap A_{\Delta_2}) & \xrightarrow{(j_{1,2*}, j_{2,1*})} & \mathcal{K}_1(A_{\Delta_1}) \oplus \mathcal{K}_1(A_{\Delta_2}) & \xrightarrow{j_{1*} - j_{2*}} & \mathcal{K}_1(A) & & \\
& & & & \downarrow \partial & & \\
\mathcal{K}_0(A) & \xleftarrow{j_{1*} - j_{2*}} & \mathcal{K}_0(A_{\Delta_1}) \oplus \mathcal{K}_0(A_{\Delta_2}) & \xleftarrow{(j_{1,2*}, j_{2,1*})} & \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2}) & & \\
\mathcal{K}_1(SA_{\Delta_1} \cap SA_{\Delta_2}) & \xrightarrow{(j_{1,2*}, j_{2,1*})} & \mathcal{K}_1(SA_{\Delta_1}) \oplus \mathcal{K}_1(SA_{\Delta_2}) & \xrightarrow{j_{1*} - j_{2*}} & \mathcal{K}_1(SA) & & \\
& & & & \downarrow \partial & & \\
\mathcal{K}_0(SA) & \xleftarrow{j_{1*} - j_{2*}} & \mathcal{K}_0(SA_{\Delta_1}) \oplus \mathcal{K}_0(SA_{\Delta_2}) & \xleftarrow{(j_{1,2*}, j_{2,1*})} & \mathcal{K}_0(SA_{\Delta_1} \cap SA_{\Delta_2}) & &
\end{array}$$

where $j_{1,2}$, $j_{2,1}$, j_1 , and j_2 are the respective inclusion maps.

Now we need to check that the ideals in Definition 5.13 satisfy the hypotheses for our controlled Mayer-Vietoris sequence. To do so, we shall make use of partitions of unity and associated multiplication operators.

Definition 5.18. Let K be a compact subset of X , let $\{U_0, \dots, U_d\}$ be a finite open cover of K , and let $\{\phi_0, \dots, \phi_d\}$ be a subordinate partition of unity. Let $s \geq 0$. For $i \in \{0, \dots, d\}$, let M_i be the multiplication operator on E_s associated to the function

$$Z_s \times X \rightarrow [0, 1], (z, x) \mapsto \sum_{g \in \Gamma} t_g(z) \phi_i(gx).$$

Lemma 5.19. With notation as above, the operators M_i have the following properties:

- (i) $\|M_i\| \leq 1$.
- (ii) If $T \in C^{*,p}(\Gamma \curvearrowright X; s)$ satisfies

$$\{x \in X : (gx, g, x) \in \text{supp}(T) \text{ for some } g \in \Gamma\} \subset K,$$
 then $T = T(M_0 + \dots + M_d)$.
- (iii) For any $T \in C^{*,p}(\Gamma \curvearrowright X; s)$ with Γ -propagation at most r , and $i \in \{0, \dots, d\}$, we have

$$\text{supp}(TM_i) \subset \left\{ (gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq s} h \cdot U_i, l(g) \leq r \right\} \cap \text{supp}(T).$$

Proof. Each M_i is a multiplication operator associated to a function taking values in $[0, 1]$ so it follows that $\|M_i\| \leq 1$.

For $i \in \{0, \dots, d\}$, $T \in C^{*,p}(\Gamma \curvearrowright X; s)$, $y, z \in P_s(\Gamma)$, and $x \in X$, we have

$$(TM_i)_{y,z}(x) = T_{y,z}(x) \cdot \sum_{h \in \Gamma} t_h(z) \phi_i(hx).$$

Hence

$$(T(M_0 + \dots + M_d))_{y,z}(x) = T_{y,z}(x) \cdot \sum_{h \in \Gamma} t_h(z) (\phi_0(hx) + \dots + \phi_d(hx)).$$

Suppose that $\{x \in X : (gx, g, x) \in \text{supp}(T) \text{ for some } g \in \Gamma\} \subset K$. If $T_{y,z}(x) \neq 0$, then $(gx, gh^{-1}, hx) \in \text{supp}(T)$ for all $g \in \text{supp}(y)$ and $h \in \text{supp}(z)$. In particular, $hx \in K$ for all $h \in \text{supp}(z)$, so

$$\sum_{h \in \Gamma} t_h(z) (\phi_0(hx) + \dots + \phi_d(hx)) = \sum_{h \in \Gamma} t_h(z) = 1,$$

and this proves (ii).

Suppose that $(gx, gk^{-1}, kx) \in \text{supp}(TM_i)$, where $T \in C^{*,p}(\Gamma \curvearrowright X; s)$ has Γ -propagation at most r . Then there exist $y, z \in P_s(\Gamma)$ with $g \in \text{supp}(y)$, $k \in \text{supp}(z)$, and $(TM_i)_{y,z}(x) \neq 0$. In particular, $T_{y,z}(x) \neq 0$, so $(gx, gk^{-1}, kx) \in \text{supp}(T)$ and $l(gk^{-1}) \leq r$. We also have $\sum_{h \in \Gamma} t_h(z) \phi_i(hx) \neq 0$, so there exists $h \in \text{supp}(z)$ with $\phi_i(hx) \neq 0$, and thus $hx \in U_i$. Since $h, k \in \text{supp}(z)$,

and $z \in P_s(\Gamma)$, we have $l(kh^{-1}) \leq s$. Now $kx = (kh^{-1})hx \in kh^{-1} \cdot U_i$, and this proves (iii). \square

Lemma 5.20. *The pair (I, J) in Definition 5.13 is a controlled Mayer-Vietoris pair for A .*

Proof. The second condition is satisfied because of the definition and the fact that I and J are ideals in A .

Let U_i be the unit space of $G_i^{+r_0}$ for $i = 0, 1$. If $a \in A_{r_0}$, then

$$K := \overline{\{x \in X : (gx, g, x) \in \text{supp}(a(t)) \text{ for some } t \in [0, \infty), g \in \Gamma\}}$$

is a compact subset of $U_0 \cup U_1$. Let M_0, M_1 be the multiplication operators defined with respect to the compact set K , the open cover $\{U_0, U_1\}$, and some choice of subordinate partition of unity $\{\phi_0, \phi_1\}$. By the previous lemma, we have $a(t)(M_0 + M_1) = a(t)$ for all t . Moreover, $\|a(t)M_i\| \leq \|a(t)\|$ for $i = 0, 1$. It remains to show that $t \mapsto a(t)M_0$ is in I_r and $t \mapsto a(t)M_1$ is in J_r for some $r \geq r_0$ (that may depend on s_0 but not on s). We will focus on the case of M_0 since the other case is similar.

Write $a = b_0 + b_1 + c$ with $b_i \in A^{s_0}(G_i^{+r_0})_{r_0}$ and $c \in A^s(G_0^{+r_0} \cap G_1^{+r_0})$. By the previous lemma, we have $\text{supp}(b_0(t)M_0) \subset \text{supp}(b_0(t))$ and also $\text{supp}(c(t)M_0) \subset \text{supp}(c(t))$ so $t \mapsto b_0(t)M_0$ is in $A^{s_0}(G_0^{+r_0})_{r_0} \subset I_{r_0}$ and $t \mapsto c(t)M_0$ is in $A^s(G_0^{+r_0} \cap G_1^{+r_0}) \subset I_{r_0}$.

Now assume that $(gx, gh^{-1}, hx) \in \text{supp}(b_1(t)M_0)$ for some t . Then there exist $y, z \in P_s(\Gamma)$ such that $g \in \text{supp}(y)$, $h \in \text{supp}(z)$, and $(b_1(t)M_0)_{y,z}(x) \neq 0$. In particular, $(b_1(t))_{y,z}(x) \neq 0$ so $y, z \in P_{s_0}(\Gamma)$ and $l(gh^{-1}) \leq r_0$. Also, $\sum_{k \in \Gamma} t_k(z)\phi_0(kx) \neq 0$ so there exists $k \in \text{supp}(z)$ such that $\phi_0(kx) \neq 0$, and thus kx is in U_0 , the unit space of $G_0^{+r_0}$. Hence

$$(gx, gh^{-1}, hx) = (gx, gk^{-1}, kx)(kx, kh^{-1}, hx) \in (G_0^{+r_0})^{+r_0} \cdot (G_0^{+r_0})^{+s_0} \subset G_0^{+(2r_0+s_0)}.$$

Hence $t \mapsto b_1(t)M_0$ is in $A^{s_0}(G_0^{+(2r_0+s_0)})_{2r_0+s_0} \subset I_{2r_0+s_0}$, and so $t \mapsto a(t)M_0$ is in $I_{2r_0+s_0}$.

Next, suppose that $a_0 \in I_{r_0}$ and $a_1 \in J_{r_0}$ such that $\|a_0 - a_1\| < \varepsilon$. Again, let U_i be the unit space of $G_i^{+r_0}$ for $i = 0, 1$. Consider

$$K_i := \overline{\{x \in X : (gx, g, x) \in \text{supp}(a_i(t)) \text{ for some } t \in [0, \infty), g \in \Gamma\}}$$

for $i = 0, 1$, and let $K = K_1 \cup K_2$, a compact subset of $U_0 \cup U_1$. Let M_0, M_1 be the multiplication operators defined with respect to the compact set K , the open cover $\{U_0, U_1\}$, and some choice of subordinate partition of unity $\{\phi_0, \phi_1\}$. Define $b(t) = a_0(t)M_1 + a_1(t)M_0$. Then $b \in I_{2r_0+s_0} \cap J_{2r_0+s_0}$, and since $a_i(t) = a_i(t)(M_0 + M_1)$ by the choice of K , we have $\|a_i(t) - b(t)\| \leq \|a_0(t) - a_1(t)\| < \varepsilon$ for $i = 0, 1$. \square

Proposition 5.21. *Let λ and P be as in Theorem 5.17 and Proposition 4.5 respectively. Let $\Gamma \curvearrowright X$ be an action. Let $r_0, s_0 \geq 0$, $d \in \mathbb{N}$, and $N \geq 1$. Then there is $r \geq \max(r_0, s_0)$ that depends only on the action, r_0 , s_0 , d , and N , and there exists $0 < \varepsilon < \frac{1}{20}$ that depends only on N with the following*

property:

Let G be an open subgroupoid of $\Gamma \ltimes X$ such that there are open subsets U_0, \dots, U_d of X with the following properties:

- (i) the unit space $G^{(0)}$ equals $\bigcup_{i=0}^d U_i$;
- (ii) for each $i \in \{0, \dots, d\}$, if G_i is the subgroupoid of $\Gamma \ltimes X$ generated by

$$\left\{ (gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot U_i, l(g) \leq r \right\},$$

then the expansion G_i^{+r} has compact closure.

Then for any $s \geq \max(r_0, s_0)$ with

$$\bigcup_{i=0}^d G_i^{+r} \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\},$$

there exists $N' \geq N$ such that the inclusion map

$$K_*^{\varepsilon, r_0, N}(A^{s_0}(G)) \rightarrow K_*^{\frac{1}{20}, s, N'}(A^s(G^{+r}))$$

is the zero map.

Proof. First consider the case $d = 0$. By assumption, the subgroupoid G' of G generated by $\{(gx, g, x) \in \Gamma \ltimes X : x, gx \in G^{(0)}, l(g) \leq r_0\}$ has compact closure. Then

$G \cap \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq r_0\} = G' \cap \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq r_0\}$
so $A^{s_0}(G)_{r_0} = A^{s_0}(G')_{r_0}$, and therefore the natural map

$$K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G')) \rightarrow K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G))$$

is the identity map. On the other hand, $A^s(G')_s = A^s(G')$ for $s \geq \max(r_0, s_0)$.

By Proposition 5.3, $K_*(A^s(G')) = 0$ so for any $x \in K_*^{\frac{1}{20P_N}, s, N'}(A^s(G'))$, there exists $N' \geq N$ such that $x = 0$ in $K_*^{\frac{1}{20}, s, N'}(A^s(G'))$. We have a commutative diagram

$$\begin{array}{ccc} K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G)) & \longrightarrow & K_*^{\frac{1}{20}, s, N'}(A^s(G)) \\ \uparrow & & \uparrow \\ K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G')) & \longrightarrow & K_*^{\frac{1}{20}, s, N'}(A^s(G')) \end{array}$$

which gives us the conclusion in the case $d = 0$.

Now suppose the result holds for some $d \in \mathbb{N}$. We want to prove the statement given $r_0, s_0, d + 1$, and N . We first consider the odd case. By Theorem 5.17, we have a control pair (λ, h) and a (λ, h) -controlled exact

Mayer-Vietoris sequence. In particular, for every $0 < \varepsilon < \frac{1}{20\lambda_N}$, $r \geq 0$, and $N \geq 1$, we have a well-defined controlled boundary map

$$\partial : K_1^{\varepsilon, r, N}(A) \rightarrow K_0^{\lambda_N \varepsilon, h_\varepsilon, N^r, \lambda_N}(I \cap J)$$

such that if $x \in K_1^{\varepsilon, r, N}(A)$ and $\partial(x) = 0$, then there exist $y \in K_1^{\lambda_N \varepsilon, h_\varepsilon, N^r, \lambda_N}(I)$ and $z \in K_1^{\lambda_N \varepsilon, h_\varepsilon, N^r, \lambda_N}(J)$ with $x = y + z$ in $K_1^{\lambda_N \varepsilon, h_\varepsilon, N^r, \lambda_N}(A)$. In the rest of the proof, we will write h_N for $h_{\frac{1}{20}, N}$.

By the induction hypothesis, there exists $r_1 \geq \max(h_N r_0, s_0)$ such that the result holds with respect to d . We will show that $r = r_1 + h_N r_0 + r_0 + s_0$ works. Let G be an open subgroupoid of $\Gamma \ltimes X$, and let U_0, \dots, U_{d+1} be open subsets of X be as in the assumptions for the $d+1$ case. Set $W_0 = \bigcup_{i=0}^d U_i$ and $W_1 = U_{d+1}$ so that $G^{(0)} = W_0 \cup W_1$. For $i = 0, 1$, let \mathcal{G}_i be the open subgroupoid of $\Gamma \ltimes X$ generated by

$$\left\{ (gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq s_0} h \cdot W_i, l(g) \leq r_0 \right\}.$$

We claim that

$$(1) \quad A^{s_0}(G)_{r_0} \subset A^{s_0}(\mathcal{G}_0)_{r_0} + A^{s_0}(\mathcal{G}_1)_{r_0}.$$

Indeed, let $a \in A^{s_0}(G)_{r_0}$ so that

$$K := \overline{\{x \in X : (gx, g, x) \in \text{supp}(a(t)) \text{ for some } t \in [0, \infty)\}}$$

is a compact subset of $G^{(0)}$. Let M_0 and M_1 be the multiplication operators defined with respect to the compact set K , the open cover $\{W_0, W_1\}$, and some choice of subordinate partition of unity. Then $a(t) = a(t)(M_0 + M_1)$ for all t . Moreover,

$$\text{supp}(a(t)M_i) \subset \left\{ (gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq s_0} h \cdot W_i, l(g) \leq r_0 \right\}$$

so $t \mapsto a(t)M_i$ is in $A^{s_0}(\mathcal{G}_i)_{r_0}$, thereby proving the claim.

Now, let $s \geq \max(r_0, s_0)$ with $\bigcup_{i=0}^{d+1} G_i^{+r} \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}$. In particular, we have $s \geq r$. For $i \in \{0, \dots, d\}$, let

$$V_i = \bigcup_{l(h) \leq h_N r_0 + s_0} h \cdot U_i,$$

which is open in X . By the definition of \mathcal{G}_0 , we see that the unit space of $\mathcal{G}_0^{+h_N r_0}$ is $\bigcup_{i=0}^d V_i$. Moreover, if H_i is the subgroupoid of $\Gamma \ltimes X$ generated by $\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r_1} h \cdot V_i, l(g) \leq r_1\}$, then since $r \geq r_1 + h_N r_0 + s_0$, the assumptions on U_0, \dots, U_{d+1} imply that each $H_i^{+r_1}$ has compact closure contained in $\{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}$. By the induction hypothesis applied to $\mathcal{G}_0^{+h_N r_0}$, there exist $0 < \varepsilon_0 < \frac{1}{20}$ and $N_0 \geq N$ such that the map

$$(2) \quad K_1^{\varepsilon_0, s, N}(A^s(\mathcal{G}_0^{+h_N r_0})) \rightarrow K_1^{\frac{1}{20}, s, N_0}(A^s(\mathcal{G}_0^{+(h_N r_0 + r_1)}))$$

is the zero map.

Let $\varepsilon_1 = \frac{\varepsilon_0}{\lambda_N P_N^2}$ and let $x \in K_1^{\varepsilon_1, r_0, N}(A^{s_0}(G))$. Let A, I , and J be defined with respect to \mathcal{G}_0 and \mathcal{G}_1 as in Definition 5.13. We may regard x as an element in $K_1^{\varepsilon_1, r_0, N}(A)$ since $A^{s_0}(G)_{r_0} \subset A^{s_0}(\mathcal{G}_0)_{r_0} + A^{s_0}(\mathcal{G}_1)_{r_0} \subset A_{r_0}$. Then $\partial(x) \in K_0^{\lambda_N \varepsilon_1, h_N r_0, \lambda_N}(I \cap J)$. But $I \cap J \cap A_{h_N r_0}$ can be identified with $A^s(\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0})$ so we may regard $\partial(x)$ as an element in $K_0^{\lambda_N \varepsilon_1, h_N r_0, \lambda_N}(A^s(\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0}))$.

Since $r \geq \max(r_0, h_N r_0, s_0)$, if we let \mathcal{G}_1^{+r} be the subgroupoid generated by

$$\left\{ (gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot W_1, l(g) \leq r \right\},$$

then we have

$$(3) \quad \mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0} \subset \mathcal{G}_1^{+h_N r_0} \subset \mathcal{G}_1^{+r} \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\},$$

the final inclusion following from the assumptions on s and W_1 . By Proposition 5.3, we have

$$K_*(A^s(\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0})) = 0.$$

Thus there exists $N_1 \geq \lambda_N$ such that $\partial(x) = 0$ in $K_0^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(I \cap J)$. Now by controlled exactness, there exist $y \in K_1^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(I)$ and $z \in K_1^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(J)$ with $x = y + z$ in $K_1^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(A)$.

Since $I_{h_N r_0} \subset A^s(\mathcal{G}_0^{+h_N r_0})_s$ and $J_{h_N r_0} \subset A^s(\mathcal{G}_1^{+h_N r_0})_s$ by the assumption on s and the observation in (3), we may regard y and z as elements in $K_1^{\lambda_N P_N \varepsilon_1, s, N_1}(A^s(\mathcal{G}_0^{+h_N r_0}))$ and $K_1^{\lambda_N P_N \varepsilon_1, s, N_1}(A^s(\mathcal{G}_1^{+h_N r_0}))$ respectively. Then there exists $N_2 \geq N_1$ such that $y = 0$ in $K_1^{\frac{1}{20}, s, N_2}(A^s(\mathcal{G}_0^{+(h_N r_0 + r_1)}))$ by (2). Since $K_*(A^s(\mathcal{G}_1^{+h_N r_0})) = 0$ by Proposition 5.3 and (3), there exists $N_3 \geq N_2$ such that $z = 0$ in $K_1^{\varepsilon_0, s, N_3}(A^s(\mathcal{G}_1^{+h_N r_0}))$.

Finally, \mathcal{G}_0 and \mathcal{G}_1 are contained in $G^{+(r_0 + s_0)}$ so each $\mathcal{G}_i^{+(h_N r_0 + r_1)}$ is contained in G^{+r} . In particular, y and z are both 0 in $K_1^{\frac{1}{20}, s, N_3}(A^s(G^{+r}))$ so $x = 0$ in $K_1^{\frac{1}{20}, s, N_3}(A^s(G^{+r}))$. This concludes the proof for the odd case.

One can check that everything from (1) onwards holds after taking suspensions throughout and using the controlled boundary map

$$\partial : K_1^{\varepsilon, r, N}(SA) \rightarrow K_0^{\lambda_N \varepsilon, h_{\varepsilon, N} r, \lambda_N}(SI \cap SJ).$$

Thus we also get the result for the even case. \square

5.3. Proof of main theorem.

Theorem 5.22. *Suppose that $\Gamma \curvearrowright X$ has finite dynamic asymptotic dimension. Then*

$$\lim_{s \rightarrow \infty} K_*(C_{L,0}^{*,P}(\Gamma \curvearrowright X; s)) = 0.$$

Thus the L_p assembly map in Definition 2.6 is an isomorphism. In other words, the L_p Baum-Connes conjecture holds for Γ with coefficients in $C(X)$

Proof. As above, we use the shorthand A^s for $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$. We need to show that for any $s_0 \geq 0$ and any $x \in K_*(A^{s_0})$, there is $s \geq s_0$ such that the map $K_*(A^{s_0}) \rightarrow K_*(A^s)$ induced by inclusion sends x to 0.

Consider the commutative diagram

$$\begin{array}{ccc} K_*^{\varepsilon, r, N}(A^{s_0}) & \longrightarrow & K_*^{\frac{1}{20}, s, N'}(A^s) \\ \downarrow c & & \downarrow c \\ K_*(A^{s_0}) & \longrightarrow & K_*(A^s) \end{array}$$

where the horizontal maps are induced by inclusion, and the vertical maps are the respective comparison maps. By Proposition 4.4, for any $0 < \varepsilon < \frac{1}{20}$, there exist $r \geq 0$ and $N \geq 1$ such that x is in the image of $c : K_*^{\varepsilon, r, N}(A^{s_0}) \rightarrow K_*(A^{s_0})$. In particular, we apply this to the ε given by Proposition 5.21. Then Proposition 5.21 and Lemma 3.13 imply that there exist $s \geq \max\{s_0, r\}$ and $N' \geq N$ such that the top horizontal map is zero. Hence the bottom horizontal map sends x to 0. \square

As mentioned in the introduction, since the left-hand side of the assembly map can be shown to be independent of p , we have the following consequence, which gives a partial answer to [13, Problem 11.2].

Corollary 5.23. *Suppose that $\Gamma \curvearrowright X$ has finite dynamic asymptotic dimension. Then the K -theory of the L_p reduced crossed product $C(X) \rtimes_{\lambda, p} \Gamma$ does not depend on p for $p \in (1, \infty)$.*

The general question of whether the K -theory of L_p reduced crossed products and L_p reduced group algebras depend on p remains open but there has been some recent work in this direction [8, 10]. This question is relevant to the usual Baum-Connes conjecture since in some cases the K -theory of these algebras may be more computable for large p so if the K -theory of these algebras is independent of p , then we get in particular a computation of the K -theory of the respective C^* -algebras.

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